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Many Boundary Representatives, One Bulk Operator: Tensor Networks, Operator-Algebra Quantum Error Correction, and the Holographic-Emergence Regime of a Reconstruction Program

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Abstract

We treat holographic emergence as one regime of a modular program in which spacetime, matter, causality, and measurement are reconstructed from invariant representation structures rather than obtained by quantizing fields on a fixed background. In this regime the invariant is compact: many boundary representatives, one logical bulk content. We take that phrase literally. A bulk effective-field-theory operator localized in the entanglement wedge of a boundary region A is not an object with independent existence that happens to have a boundary description; it is defined as the equivalence class of boundary operators $\{O_A\}$ related by the code's logical-equivalence relation, one representative for each region A whose wedge contains the bulk point. We assemble the finite-dimensional mathematics that makes this precise: an isometric code $V : \mathcal{H}_{\text{code}} \to \mathcal{H}_{\text{phys}}$ with $V^\dagger V = \mathbb{1}$; perfect tensors and absolutely maximally entangled states as its building blocks; stabilizer codes as a computable source of perfect tensors; operator-algebra quantum error correction (OAQEC) with its correctable von Neumann subalgebras; and complementary recovery, the structure that makes entanglement-wedge reconstruction consistent. We prove the elementary facts in full (Knill–Laflamme correctability of a region's complement $\Rightarrow$ bulk reconstruction on the region; a perfect tensor $\Rightarrow$ operator pushing and one-sided recovery; a discrete Ryu–Takayanagi min-cut on a tree tensor network) and state the deeper theorems of Almheiri–Dong–Harlow, Harlow, and Jafferis–Lewkowycz–Maldacena–Suh with the epistemic labels they deserve (standard mathematics S, physical heuristic H, speculative P). We work a small holographic pentagon code explicitly, pushing a single bulk logical operator to three-body boundary representatives on complementary regions and computing the min-cut entropy on a toy graph. We place the regime inside a weakest-link status calculus and compose it, honestly, with three sibling regimes: BV–BRST gauge redundancy (where the surface-code $H_1(\Sigma; \mathbb{Z}_2)$ identification is the one place "gauge redundancy = logical redundancy" is genuinely S), factorization-algebra locality, and emergent Lorentzian geometry. We state plainly what is not settled: exact codes are finite-dimensional and isometric while real AdS/CFT is approximate, with gravitational backreaction and type III boundary algebras; and no classification theorem exists for which gauge redundancies behave like logical-code redundancies beyond the surface-code case. The composite status of the regime is H, set by the entanglement-wedge-reconstruction entry, not by the toy-model mathematics, which is S/H.

Contents

1 Introduction

1.1 The reconstruction thesis, and this paper's regime

The wider project this paper belongs to organizes itself around one sentence. *Quantum gravity is not primarily the quantization of material objects in spacetime; it is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures.* Read that way, a bulk spacetime is not a substrate one begins with and then quantizes. It is an output: the semiclassical shadow that a deeper representation—a spin network, an asymptotically safe renormalization-group trajectory, a celestial correlator, or, here, a tensor-network holographic code—must reproduce in an appropriate limit.

Among the twelve regimes of the program, the holographic-emergence regime is the one where the reconstruction thesis reaches its *measurement* clause most directly. In the classical-geometry regime one reconstructs a metric; in the spin-network regime one reconstructs spatial geometry from $SU(2)$ representation labels. Here the object reconstructed is not merely geometry but the very act of observing a bulk operator. A bulk observable is only ever accessed through a boundary measurement, and there is no canonical boundary measurement: there is a redundant family of them, one per boundary region whose entanglement wedge contains the bulk point. The invariant is the equivalence class, not any single representative. This is what the slogan "many boundary representatives, one logical bulk content" names, and quantum error correction is the mathematics that makes it exact.

The connection is due to Almheiri, Dong, and Harlow [1], who observed that the redundancy of the AdS/CFT map—a bulk operator deep in the interior has many inequivalent-looking boundary descriptions, and the boundary region needed to reconstruct it depends on where it sits—is exactly the redundancy of a quantum error-correcting code, in which a logical qubit is encoded nonlocally into physical qubits so that it survives erasure of a subset of them. Pastawski, Yoshida, Harlow, and Preskill [2] then gave the phenomenon a solvable toy model: the HaPPY code, a tensor network of perfect tensors tiling the hyperbolic plane, which is an exact isometry, reproduces the Ryu–Takayanagi entropy formula in a discrete min-cut form, and has the negative tripartite information characteristic of holographic states. Harlow

[3] recast the whole package in the language of operator-algebra quantum error correction, proving that entanglement-wedge reconstruction, the Jafferis–Lewkowycz–Maldacena–Suh relative-entropy equality [4], and the Ryu–Takayanagi formula with an area operator are three faces of one code-theoretic structure.

1.2 What this paper does, and what it does not claim

This paper is a self-contained, epistemically honest account of the holographic emergence regime, written to sit inside the program's modular library. It does three things.

First, it fixes the finite-dimensional mathematics precisely enough to prove the elementary statements in full and to state the deep ones without inflation. The elementary statements—that an isometric encoding whose complement is Knill–Laflamme correctable reconstructs its logical algebra on a region, that a perfect tensor gives operator pushing and one-sided recovery, that a tree tensor network computes a min-cut entropy—are genuinely standard mathematics, S. The deep statements—that this is *the* correct dictionary for a general quantum-gravity bulk beyond the toy regime—are heuristic, H, and we label them so.

Second, it works a concrete example: a single holographic pentagon (the $[[5,1,3]]$ perfect-tensor code) and a small patch, with the bulk logical operator pushed explicitly to boundary representatives, complementary recovery exhibited on two disjoint boundary regions, and a discrete Ryu–Takayanagi cut computed on a toy graph. Every claim in the example is checkable by hand, and the accompanying Haskell code checks it mechanically.

Third, it composes the regime with its neighbors under the program's weakest-link status calculus, without ever asserting a monolithic unification. The sharpest such composition is with the BV–BRST gauge-redundancy regime: the toric/surface code's identification of logical operators with $\mathrm{H}_1(\Sigma;\mathbb{Z}_2)$ classes [11, 12] is the one place in the whole library where "gauge redundancy equals logical redundancy" is a proved, status-S fact rather than a slogan, and we present it as the concrete testbed for the roadmap's open problem 5.

We are explicit throughout about the gap between the toy models and real AdS/CFT, and about the entirely separate character of celestial/flat-space holography, which this regime must not be silently merged with.

1.3 The seed dictionary

Every paper in the program opens with the same translation table, the shared semantic contract between its regimes. Table 1 reproduces the four rows most relevant to this paper, each with its epistemic status. The two rows in the middle carry this paper's own dictionary entries; the composite status of the regime is set by the weaker of them.

1.4 Notation and standing conventions

All Hilbert spaces are finite-dimensional over $\mathbb{C}$ unless stated otherwise; this is the honest home of the exact statements. We write $\mathcal{B}(\mathcal{H})$ for the bounded operators on $\mathcal{H}$ (all operators, in finite dimension), $\mathbb{1}$ for the identity, Tr for the trace, and A' for the commutant of a subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$. A *code* is an isometry $V : \mathcal{H}_{\text{code}} \to \mathcal{H}_{\text{phys}}$, $V^\dagger V = \mathbb{1}_{\mathcal{H}_{\text{code}}}$, with code projector $P = VV^\dagger$ and code subspace $\mathcal{H}_C = \operatorname{im} V = P\mathcal{H}_{\text{phys}}$. We call $\mathcal{H}_{\text{code}} \cong \mathcal{H}_{\text{bulk}}$ the bulk

Status	Mathematics	Physical representation	Translation
S	Gauge groupoid / quotient stack	many representatives, one observable content	isotropy-preserving quotient
S/H	Tensor network / isometric encoding	holographic bulk–boundary map	encoding realization
H	Entanglement-wedge reconstruction map	bulk observable represented on a boundary subregion	logical reconstruction
S/H	Area / entropy functional and entanglement structure	holographic entropy observable	entropy geometrization

Table 1: The dictionary rows this paper realizes, with the program's epistemic labels. "Tensor network / isometric encoding" and the "area / entropy functional" are S/H: the toy-model mathematics is solid, the claim that it is the physical bulk dictionary is heuristic. "Entanglement-wedge reconstruction map" is H; it is the weakest link and sets the composite status of the regime.

(logical) space and $\mathcal{H}_{\text{phys}} \cong \mathcal{H}_{\text{bdy}}$ the boundary (physical) space. A bipartition of the physical system into subsystems A and its complement A^c induces a factorization $\mathcal{H}_{\text{phys}} \cong \mathcal{H}_A \otimes \mathcal{H}_{A^c}$ when A is a set of physical qubits, and, more generally, a pair of commuting von Neumann subalgebras $(\mathcal{M}_A, \mathcal{M}_{A^c})$. The n-qubit Pauli group is P_n; a stabilizer group $S \subseteq \mathsf{P}_n$ is abelian with $-\mathbb{1} \notin S$. Section-level cross-references use Section 1-style labels. Epistemic tags **[S]**, **[H]**, **[P]** (and composites **[S/H]**, **[H/P]**) annotate every labeled result.

2 The representation-stack core and the status calculus

Before the physics we recall the two structural devices the program reuses in every regime: the representation stack (which says physical content is a groupoid, not a set) and the S/H/P status calculus (which bounds how regimes compose). Both specialize cleanly to holographic codes.

2.1 The representation stack

A *representation entry* is a tuple $E = (M, P, \tau, \sigma)$: a mathematical object M, a physical target P, a translation datum τ, and an epistemic status $\sigma \in \{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$. A *representation prestack* is a pseudofunctor $\mathrm{Real} : \mathbf{Dom}^{\mathrm{op}} \to \mathbf{Gpd}$ into groupoids; it is a *representation stack* for a Grothendieck topology on **Dom** when compatible local entries glue to global entries uniquely up to coherent isomorphism. The content of "stack, not prestack" is that gluing is by *equivalence*, not equality, so automorphism data—gauge redundancy, stabilizers, code redundancy—is retained rather than erased. The single structural fact threading all twelve regimes of the program is:

> *Physical content is a groupoid (a stack), not a coarse quotient set: the automorphisms* $\mathrm{Aut}_{[X/G]}(x) \cong \mathrm{Stab}_G(x)$ *are part of the data.*

In the holographic-code regime this fact appears as follows. The physical content of "a bulk operator" is not a single boundary operator (a point of a quotient set) but the groupoid whose objects are the boundary representatives O_A and whose morphisms are the code's logical-equivalence relations $O_A \sim O_{A'}$ whenever both regions' wedges contain the bulk point. The redundancy—the many representatives—is exactly the automorphism data the stack retains. We make this precise in Sections 4 and 5.

2.2 The S/H/P status calculus

There are three ordered warrants $\mathsf{S} < \mathsf{H} < \mathsf{P}$: S is a standard mathematical or mathematical-physics correspondence, H a strong heuristic physical representation, and P a speculative ontological extension. A *status* is a nonempty set of warrants, displayed by its range (S/H, H/P), whose governing datum is its worst (largest) component.

Definition 2.1 (Status composition)**.** For statuses a, b with worst components $w(a), w(b)$, the composite is

$$\mathrm{compose}(a, b) \;=\; \{\; \max\big(w(a), w(b)\big)\;\},$$

the single dominant warrant. A single regime is summarized by the *range* (union) of its entries' warrants; a *chain* across regimes is summarized by compose, which collapses to the weakest link.

Proposition 2.2 (Monotonicity, unit, associativity)**. [S]** *The operation* compose *is monotone ($w(\mathrm{compose}(a, b)) \geq w(a)$ and $\geq w(b)$), has* S *as a two-sided unit (*$\mathrm{compose}(\mathsf{S}, a) = \{w(a)\}$*), and is associative. Consequently no chain of regimes has a better status than its weakest constituent claim.*

Proof. max on the linearly ordered set $\{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$ is idempotent, commutative, associative, and monotone, with least element S as unit. The three assertions are these facts read off the worst component. The final sentence is monotonicity applied inductively along a chain. □

This is the formal engine of Section 7. It is proved as a Lean4 stub in the program's formalization layer and checked as an executable property in the Haskell code accompanying this paper (Section 9). We stress its discipline: composing two S/H regimes does not automatically yield S/H; one composes the *specific claims* used in the bridge, and the bridge may be weaker than either endpoint.

3 Codes, isometries, and correctability

We now build the regime's mathematics from the ground up. Everything in this section is finite-dimensional and status-S; it is the solid floor on which the heuristic bulk interpretation later stands.

3.1 Isometric encodings

Definition 3.1 (Isometric code)**.** An *isometric code* is a linear isometry $V : \mathcal{H}_{\mathrm{code}} \to \mathcal{H}_{\mathrm{phys}}$, that is $V^\dagger V = \mathbb{1}_{\mathcal{H}_{\mathrm{code}}}$. Its *code projector* is $P = VV^\dagger$, an orthogonal projector onto the *code subspace* $\mathcal{H}_C = \operatorname{im} V$. Fix a *code algebra* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{code}})$: a von Neumann subalgebra of the bulk operators, with center $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$. The *encoding map* on operators is $\hat{O} \mapsto V\hat{O}V^\dagger$ for $\hat{O} \in \mathcal{A}$, and the *logical algebra* is its image $\mathcal{A}_L = \{V\hat{O}V^\dagger : \hat{O} \in \mathcal{A}\} \subseteq \mathcal{B}(\mathcal{H}_C) = P\mathcal{B}(\mathcal{H}_{\mathrm{phys}})P$. The maximal choice $\mathcal{A} = \mathcal{B}(\mathcal{H}_{\mathrm{code}})$ (a full matrix algebra) has trivial center $\mathbb{C}\mathbb{1}$ and gives a *subsystem* code; a proper subalgebra $\mathcal{A}$ with nontrivial center $\mathcal{Z}(\mathcal{A})$ gives a *subalgebra* code, and it is the center that will carry the superselected area operator of Theorem 4.2. Unless stated otherwise "bulk operator" means an element of the chosen $\mathcal{A}$.

The isometry condition $V^\dagger V = \mathbb{1}$ says the encoding loses no logical information: distinct logical states map to distinguishable (orthonormal, if the inputs were) physical states. The complementary condition $VV^\dagger = P \neq \mathbb{1}$ (unless $\dim \mathcal{H}_{\mathrm{code}} = \dim \mathcal{H}_{\mathrm{phys}}$) says the physical space is larger, and the extra room is what redundancy lives in. A bulk operator $\hat{O}$ has boundary representative $V\hat{O}V^\dagger$; the reconstruction problem is whether this global boundary operator is equal, *on the code subspace*, to an operator supported on a proper region.

Definition 3.2 (Reconstruction on a region)**.** Let A be a physical subsystem, $\mathcal{H}_{\mathrm{phys}} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$. A bulk operator $\hat{O} \in \mathcal{B}(\mathcal{H}_{\mathrm{code}})$ is *reconstructable on A* if there is an operator $O_A \in \mathcal{B}(\mathcal{H}_A) \otimes \mathbb{1}_{A^c}$ with

$$O_A\, V \;=\; V\,\hat{O} \qquad \text{equivalently} \qquad O_A\, P \;=\; P\,(V\hat{O}V^\dagger) \;=\; (V\hat{O}V^\dagger).$$

The operator O_A is a *boundary representative* of $\hat{O}$ on A. It agrees with the global encoded operator on the code subspace but may differ arbitrarily off it.

The phrase "may differ arbitrarily off it" is the crux. Two boundary representatives O_A and $O_{A'}$ of the same bulk operator on different regions need not be equal as physical operators; they are equal only after multiplication by P. This is precisely the equivalence relation of Section 2.1: the bulk operator is the class, the representatives are its objects.

3.2 The Knill–Laflamme conditions

The correctability of a code against a set of errors is governed by a single algebraic condition.

Definition 3.3 (Correctable error set)**.** A set $\{E_a\} \subseteq \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ of *errors* is *correctable* for the code V if there is a recovery channel $\mathcal{R} : \mathcal{B}(\mathcal{H}_{\mathrm{phys}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{phys}})$ (a completely positive trace-preserving map) with $\mathcal{R}\big(E_a\, V\rho V^\dagger\, E_b^\dagger\big) \propto V\rho V^\dagger$ for all code states ρ and all a, b, up to a constant matrix.

Theorem 3.4 (Knill–Laflamme conditions)**. [S]** *The error set $\{E_a\}$ is correctable for the code with projector P if and only if there is a Hermitian matrix (c_{ab}) with*

$$P\, E_a^\dagger E_b\, P \;=\; c_{ab}\, P \qquad \textit{for all } a, b. \tag{1}$$

Proof sketch. This is the theorem of Knill and Laflamme [8]; we recall the structure. Necessity: if a recovery $\mathcal{R}$ exists, its Kraus operators R_k satisfy $R_k E_a P = \lambda_{ka} P$ for scalars λ_{ka} (recovery must return code states to code states independent of the error), and $P E_a^\dagger E_b P = \sum_k \bar{\lambda}_{ka} \lambda_{kb} P = c_{ab} P$. Sufficiency: diagonalize $(c_{ab}) = u^\dagger d u$; the operators $F_\alpha = \sum_a u_{\alpha a} E_a$ satisfy $P F_\alpha^\dagger F_\beta P = d_\alpha \delta_{\alpha\beta} P$, so the nonzero F_α map the code subspace to mutually orthogonal subspaces isometrically (after normalization), and the recovery channel projects onto each and rotates back. □

The physical reading of (1) is that no error in the set leaves any detectable, logical-state-dependent imprint on the code subspace: $P E_a^\dagger E_b P$ is a state-independent constant times P. In the holographic setting the errors of interest are all operators supported on the complementary region A^c; if those are correctable, the erasure of A^c can be undone, which is the same as saying the logical information is reconstructable from A.

Proposition 3.5 (Isometry plus Knill–Laflamme gives reconstruction). **[S]** *Let V be an isometric code and A a physical subsystem. Suppose every operator $X \in \mathcal{B}(\mathcal{H}_{A^c}) \otimes \mathbb{1}_A$ supported on the complement is correctable, i.e. $P X^\dagger Y P = c_{XY} P$ for a bilinear form c on the complement algebra. Then every bulk operator $\hat{O} \in \mathcal{B}(\mathcal{H}_{\text{code}})$ is reconstructable on A in the sense of Theorem 3.2.*

Proof. Correctability of the entire complement algebra is the statement that the complement acts trivially on logical data: for all X supported on A^c, $PXP = c(X)\, P$ with c a state (a positive linear functional). Fix a bulk operator $\hat{O}$ and set $\tilde{O} = V \hat{O} V^\dagger \in \mathcal{B}(\mathcal{H}_C)$. Consider the conditional expectation onto operators supported on A obtained by tracing out A^c against the maximally mixed state and restricting to the code subspace; call it Π_A. By the correctability hypothesis, for any code state ρ and any X on A^c, $\mathrm{Tr}(\tilde{O}\, X \rho)$ depends on X only through $c(X)$, so $\tilde{O}$ commutes with the complement algebra on the code subspace. By the double commutant / structure theorem for a factor bipartition, an operator commuting with $\mathcal{B}(\mathcal{H}_{A^c})$ on the code subspace is represented on $\mathcal{B}(\mathcal{H}_A)$ there; choosing $O_A = \Pi_A(\tilde{O})$ extended arbitrarily off the code subspace gives $O_A P = \tilde{O}$, which is Theorem 3.2. Uniqueness of the representative holds only modulo P, exactly the equivalence class of Section 2.1. □

Theorem 3.5 is the finite-dimensional skeleton of entanglement-wedge reconstruction. What upgrades it from "correctable complement" to "the wedge is exactly the reconstructible region" is the more refined operator-algebra statement of Sections 4 and 5; the point of stating the skeleton first is that this part is entirely standard, provable in two paragraphs, and status-S.

3.3 Stabilizer codes as a source of examples

Perfect tensors, the building blocks of the HaPPY code, are most easily produced as stabilizer states, so we recall the stabilizer formalism.

Definition 3.6 (Stabilizer code). Let P_n be the n-qubit Pauli group. A *stabilizer group* is an abelian subgroup $S \subseteq \mathsf{P}_n$ with $-\mathbb{1} \notin S$. Its *code subspace* is $\mathcal{H}_S = \{|\psi\rangle : g\,|\psi\rangle = |\psi\rangle \ \forall g \in S\}$. If S has $n-k$ independent generators, $\dim \mathcal{H}_S = 2^k$: an $[[n,k]]$ code. *Logical operators* are elements of the normalizer $\mathrm{N}(S)$ modulo S; the *code distance* is the minimum weight of an element of $\mathrm{N}(S) \setminus S$.

Proposition 3.7 (Stabilizer codes satisfy Knill–Laflamme). **[S]** *A stabilizer code with distance d corrects every Pauli error of weight $\leq \lfloor (d-1)/2 \rfloor$, and detects erasure of any $d-1$ qubits. For an erasure of a region A^c with $|A^c| < d$, all logical operators are reconstructable on A.*

Proof. For Pauli errors E_a, E_b, the product $E_a^\dagger E_b$ is (up to phase) a Pauli operator g. If $g \in S$ then $PgP = P$; if $g \in \mathsf{P}_n \setminus \mathrm{N}(S)$ then g anticommutes with some stabilizer and $PgP = 0$; the intermediate case $g \in \mathrm{N}(S) \setminus S$ is excluded when $\mathrm{wt}(g) < d$. Thus $PE_a^\dagger E_b P \in \{0, P\} \subseteq \mathbb{C}P$, which is (1). The distance/erasure statement is the special case where errors are all Paulis supported on A^c: if $|A^c| < d$ no element of $\mathrm{N}(S) \setminus S$ is supported there, so the hypothesis of Theorem 3.5 holds. □

Example 3.8 (The three-qutrit code). The minimal exact example, due to Almheiri–Dong–Harlow [1] after Cleve–Gottesman–Lo, encodes one qutrit into three so that the logical operator is reconstructable on *any two* of the three physical qutrits but on *no single* qutrit. The three-dimensional code space has orthonormal logical basis

$$\begin{aligned}
|\bar{0}\rangle &= \tfrac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle),\\
|\bar{1}\rangle &= \tfrac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle),\\
|\bar{2}\rangle &= \tfrac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle),
\end{aligned}$$

i.e. the three states $\frac{1}{\sqrt{3}} \sum_j |j,\, j+a,\, j+2a\rangle$ with additions mod 3 and $a = 0, 1, 2$. (Note $|\bar{0}\rangle$ alone does not span the code space: it is fixed by the cyclic qutrit shift $X^{\otimes 3}$ and by the simultaneous translation of all three registers.) Any one qutrit is maximally mixed (carries no logical information); any two determine the logical qutrit completely. This is the smallest system exhibiting the holographic pattern: reconstruction on a region iff the region is large enough, with complementary regions never both failing.

3.4 Perfect tensors and absolutely maximally entangled states

Definition 3.9 (Perfect tensor). A tensor $T_{i_1 \dots i_N}$ with N legs of bond dimension χ is *perfect* if, for every bipartition of the legs into a set of size $m \leq \lfloor N/2 \rfloor$ and its complement, the map from the smaller set to the larger is proportional to an isometry. For even $N = 2k$ this is the requirement that every balanced k-vs-k grouping is (proportional to) an isometry; equivalently the state $|T\rangle = \sum T_{i_1 \dots i_N} |i_1 \dots i_N\rangle$ is *absolutely maximally entangled*: every reduction to at most $\lfloor N/2 \rfloor$ of the N parties is maximally mixed. For odd $N = 2k+1$ the condition is that every grouping of k legs maps isometrically into the complementary $k+1$; the three-leg GHZ-type tensor $T_{ijl} \propto \delta_{ij}\delta_{jl}$ (an isometry from any one leg to the other two) is the smallest example and is the one used in the tree of Section 6.3.

Proposition 3.10 (Perfect tensor gives operator pushing and one-sided recovery). **[S]** *Let T be a perfect tensor on N legs, one distinguished as a bulk (dangling) leg and $N-1$ as boundary legs, and view T as the encoding isometry $V : \mathcal{H}_{\mathrm{bulk}} \to \mathcal{H}_{\mathrm{bdy}}^{\otimes (N-1)}$. Then:*

(a) *A logical operator on the bulk leg can be "pushed" to a boundary operator supported on any set A of boundary legs with $|A| \geq \lceil N/2 \rceil$, i.e. $V\hat{O} = O_A V$ with O_A supported on A.*

(b) *For a bipartition* (A, A^c) *of the boundary legs, the bulk leg is reconstructable on the larger side, and* at most one *side reconstructs it; the code has complementary recovery for* (A, A^c) *in the trivial (single-block) sense of Theorem 4.2, with the bulk algebra carried entirely by the larger side and the identity algebra by the smaller.*

Proof. (a) Group the bulk leg with the boundary legs of A^c on one side; if $|A| \geq \lceil N/2 \rceil$ then the opposite side has $\leq \lfloor N/2 \rfloor$ legs, so by perfectness the map from the A^c-plus-bulk side into A is an isometry, and a logical operator inserted on the bulk leg is pushed through it to an operator O_A supported on A: $V\hat{O} = O_A V$. (b) No-cloning forbids both A and A^c from reconstructing the same nontrivial bulk algebra, since their operators would then have to agree on the code subspace while commuting as operators on disjoint regions; only the side of size $\geq \lceil N/2 \rceil$ reconstructs the bulk leg, and the smaller side reconstructs nothing. This is exactly the single-block case of (2): one block, the bulk factor on the larger region, the trivial (one-dimensional) factor on the smaller. Genuine two-sided complementary recovery, with both sides reconstructing *different* bulk legs, requires a network with more than one bulk leg, exhibited in Section 6.2. □

The operational content of Theorem 3.10 is the "operator pushing" rule that makes the HaPPY code computable: a bulk operator is dragged across perfect tensors one at a time, each time being replaced by an operator on the tensor's other legs, until it lands on the boundary. We use exactly this in the worked example of Section 6.

Example 3.11 (The [[5, 1, 3]] code as a six-leg perfect tensor)**.** The five-qubit code [[5, 1, 3]] with stabilizer generated by the cyclic shifts of $XZZXI$ encodes one logical qubit into five physical qubits with distance three. Its encoding isometry, together with the logical leg, forms a six-leg perfect tensor of bond dimension two: every three-vs-three split is an isometry. This is the elementary cell of the HaPPY pentagon code. Because it is a stabilizer state, all operator-pushing rules are computable by Pauli conjugation, which is what the accompanying code does.

4 Operator-algebra quantum error correction

Entanglement-wedge reconstruction is not a subsystem statement ("qubit A carries the logical qubit") but a subalgebra statement ("the algebra of the wedge is represented in the algebra of A"). The right language is operator-algebra quantum error correction, OAQEC, due to Bény–Kempf–Kribs [9] and, in the holographic form, Harlow [3].

4.1 Correctable subalgebras

Definition 4.1 (Correctable subalgebra)**.** Let $V : \mathcal{H}_{\text{code}} \to \mathcal{H}_{\text{phys}}$ be a code and $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H}_{\text{phys}})$ a von Neumann subalgebra. Say $\mathcal{M}$ is *correctable for the complementary erasure* A^c (or, loosely, "$\mathcal{M}$ is reconstructable on A") if every element of the encoded logical algebra that is represented by $\mathcal{M}$ can be recovered after tracing out A^c; formally, there is a channel $\mathcal{R}_A$ with $\mathcal{R}_A \circ \mathrm{Tr}_{A^c} \circ (V \cdot V^\dagger) = (V \cdot V^\dagger)$ on the $\mathcal{M}$-part of the logical algebra.

Definition 4.2 (Complementary recovery)**.** A code has *complementary recovery* for a bipartition (A, A^c) if the boundary algebras $(\mathcal{M}_A, \mathcal{M}_{A^c}) = (\mathcal{B}(\mathcal{H}_A) \otimes \mathbb{1},\ \mathbb{1} \otimes \mathcal{B}(\mathcal{H}_{A^c}))$ are in commutant position and there exist *bulk* algebras $(\mathcal{A}_a, \mathcal{A}_{a^c})$ with $\mathcal{A}_a$ represented in $\mathcal{M}_A$ and $\mathcal{A}_{a^c}$ represented in $\mathcal{M}_{A^c}$, such that $\mathcal{A}_a$ and $\mathcal{A}_{a^c}$ generate the full logical algebra and their intersection is the *center*, generated by the *area operator*. Explicitly there is a decomposition

$$\mathcal{H}_C \ \cong\ \bigoplus_\alpha \mathcal{H}_a^{(\alpha)} \otimes \mathcal{H}_{a^c}^{(\alpha)}, \tag{2}$$

block-diagonal in the eigenspaces of the area operator, with $\mathcal{A}_a$ acting on the first factor and $\mathcal{A}_{a^c}$ on the second within each block α.

The block label α in (2) is the eigenvalue of the area operator, an element of the center $\mathcal{Z} = \mathcal{A}_a \cap \mathcal{A}_{a^c}$. This is the exact finite-dimensional shadow of the statement that the Ryu–Takayanagi surface's area is a superselected, classical quantity: it commutes with all wedge operators on both sides.

4.2 The reconstruction theorem

We now state the central theorem of the regime. We separate its parts by status.

Theorem 4.3 (Complementary recovery and the area operator; Harlow)**.** **[S/H]** *Let $V : \mathcal{H}_{\text{code}} \to \mathcal{H}_{\text{phys}}$ be a code and (A, A^c) a bipartition. The following are equivalent as finite-dimensional statements:*

(i) *The code has complementary recovery for (A, A^c), i.e. the decomposition* (2) *holds.*

(ii) *Every logical operator in the "wedge algebra of A" $\mathcal{A}_a$ is reconstructable on A, and every logical operator in $\mathcal{A}_{a^c}$ is reconstructable on A^c.*

(iii) *For all code states ρ, σ, the boundary relative entropy on A equals the bulk relative entropy of the wedge,*

$$S^A_{\text{rel}}(\rho\|\sigma) \ =\ S^a_{\text{rel}}(\rho\|\sigma), \tag{3}$$

the Jafferis–Lewkowycz–Maldacena–Suh equality. Equivalently there is an operator identity of modular Hamiltonians $H_A^{\text{mod}} = \hat{\mathcal{A}}_A/4G_N + H_a^{\text{mod}}$ carrying the central area operator $\hat{\mathcal{A}}_A$; the area term enters the von Neumann entropy $S(\rho_A) = \langle \hat{\mathcal{A}}_A \rangle_\rho / 4G_N + S_{\text{bulk}}(\rho_a)$ (the Ryu–Takayanagi formula) but cancels *in the relative entropy, since it appears identically in $-\operatorname{Tr}(\rho_A \log \sigma_A)$ and in $S(\rho_A)$ and drops out of their difference. This is why* (3) *has no explicit area term while the entropy* (4) *does.*

The equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) *is standard operator algebra (*S*); the identification of the center element $\hat{\mathcal{A}}_A$ with a geometric area* $\text{Area}(\gamma_A)/4G_N$ *is the heuristic bulk-geometry input (*H*), which is why the composite is* S/H.

Proof sketch and attribution. The equivalence of (i), (ii) and the operator form of (iii) is Harlow's OAQEC theorem [3], which extends the subsystem statement of Almheiri–Dong–Harlow [1] to subalgebra codes. Given the block decomposition (2), define $\hat{\mathcal{A}}_A$ to act as

$4G_N \log(\dim \mathcal{H}_{a^c}^{(\alpha)} / \dim \mathcal{H}_a^{(\alpha)})$-type constants on block α (the precise expression involves the block dimensions and the reduced code state); it is central by construction. Reconstructability of $\mathcal{A}_a$ on A follows because within each block the wedge algebra acts only on the first factor, which is carried by A. The relative-entropy equality (3) is the Jafferis–Lewkowycz–Maldacena–Suh statement [4]; its equivalence to (i) in finite dimension is the content of Harlow's Theorem. The geometric reading of the center—that $\hat{\mathcal{A}}_A$ is the area of a minimal surface—is an *input* from AdS/CFT, not a theorem of the code; the code only guarantees that *some* central element plays this role. □

Remark 4.4 (Why the status is not pure S). **[H]** Theorem 4.3 is a genuine theorem about codes. What is H, not S, is the physics dictionary: that the center element is the area of a bulk minimal surface, that the "bulk algebra" $\mathcal{A}_a$ is the algebra of a semiclassical effective field theory in a spacetime region, and that finite-dimensional codes model the genuinely infinite-dimensional, type-III boundary algebras of a continuum CFT. Each of these is well-motivated and, in the toy models, exactly realized; none is a theorem about quantum gravity. We keep the mathematics and the dictionary on separate lines throughout.

4.3 Approximate recovery and the Petz map

Real holography is not an exact code. Relative entropies are only approximately preserved, and the clean equivalences of Theorem 4.3 become approximate. The tool that survives is a *universal* recovery channel.

Definition 4.5 (Petz and twirled Petz maps)**.** For a channel $\mathcal{N}$ (here the erasure Tr_{A^c}) and a reference state σ on the code space, the *Petz map* $\mathrm{Petz}_{\sigma,\mathcal{N}}$ sends physical (image) operators back to code operators by

$$\mathrm{Petz}_{\sigma,\mathcal{N}}(\cdot) \;=\; \sigma^{1/2}\, \mathcal{N}^{\dagger}\big(\mathcal{N}(\sigma)^{-1/2}(\cdot)\mathcal{N}(\sigma)^{-1/2}\big)\, \sigma^{1/2}.$$

The *twirled Petz map* is its rotation-averaged version

$$\mathcal{R}_{\sigma}(\cdot) \;=\; \int_{-\infty}^{\infty} dt\; \beta_0(t)\; \sigma^{-it}\; \mathrm{Petz}_{\sigma,\mathcal{N}}\big(\mathcal{N}(\sigma)^{it}(\cdot)\,\mathcal{N}(\sigma)^{-it}\big)\, \sigma^{it},$$

where the inner conjugation is by the modular flow $\mathcal{N}(\sigma)^{it}$ of the *physical* reference state $\mathcal{N}(\sigma)$ (so the argument stays in the physical space) and the outer conjugation by the modular flow σ^{it} of the *code* state, with the standard weight $\beta_0(t) = \frac{\pi}{2}(\cosh \pi t + 1)^{-1}$.

Proposition 4.6 (Universal recovery bound; Cotler et al.)**.** **[S/H]** *The twirled Petz map is a* universal *recovery channel: its reconstruction fidelity is controlled by the loss of relative entropy under the erasure channel. Consequently, when the JLMS equality* (3) *holds only up to a deficit δ, the entanglement wedge of A is reconstructable on A up to an error bounded by a function of δ that vanishes as $\delta \to 0$.*

Attribution. This is the theorem of Cotler, Hayden, Penington, Salton, Swingle, and Walter [7], building on the universal-recovery inequalities of Junge–Renner–Sutter–Wilde–Winter and the exact-recovery characterization via the Petz map. The finite-dimensional statement—that a subalgebra is correctable iff the twirled Petz map recovers it, with error controlled by the conditional mutual information / relative-entropy deficit—is standard quantum information theory (S). Its application to the entanglement wedge, where the relevant approximate equality is the bulk-effective-field-theory JLMS relation, is the H input, so the composite is S/H. □

The upshot is that "many boundary representatives, one bulk operator" is robust: one does not need the idealized exact code to have a well-defined reconstruction, a controlled approximate one suffices, and the control is quantitative.

5 Entanglement wedges and the Ryu–Takayanagi functional

We now connect the algebraic picture to geometry: which region is the "wedge," and why the center element deserves to be called an area.

5.1 The Ryu–Takayanagi and HRT surfaces

Definition 5.1 (RT/HRT surface and entanglement wedge)**.** For a boundary region A in a holographic state, the *Ryu–Takayanagi surface* γ_A is the minimal-area bulk codimension-two surface homologous to A (anchored on ∂A); in time-dependent states it is the Hubeny–Rangamani–Takayanagi *extremal* surface. The *entanglement wedge* $\mathcal{E}(A)$ is the bulk domain of dependence of a bulk Cauchy slice bounded by A and γ_A.

Theorem 5.2 (Ryu–Takayanagi entropy formula)**. [H]** *In the semiclassical holographic regime, the entanglement entropy of a boundary region A is*

$$S(A) \;=\; \frac{\mathrm{Area}(\gamma_A)}{4G_N} \;+\; S_{\mathrm{bulk}}\big(\mathcal{E}(A)\big) \;+\; \cdots\,, \tag{4}$$

the area of the minimal surface in Planck units plus the bulk entanglement entropy of the wedge, with subleading quantum corrections.

The formula (4) is the Ryu–Takayanagi proposal [5], in the quantum-corrected Faulkner–Lewkowycz–Maldacena form for the bulk term and the covariant HRT form [6] for time dependence. It is H: derived in AdS/CFT under semiclassical assumptions, not proved as pure mathematics. Its importance here is structural. The additive split—area (a geometric, state-independent number, at leading order) plus bulk entropy (a genuinely quantum, state-dependent term)—is exactly the split (3) of Theorem 4.3 between the center element $\hat{\mathcal{A}}_A$ and the bulk modular data. The holographic code makes this split exact and finite-dimensional.

5.2 Discrete Ryu–Takayanagi from a tensor network

On a tensor network the area term becomes a combinatorial minimal cut, and this part is provable.

Theorem 5.3 (Min-cut bound on a tensor network)**.** **[S]** *Let $|\Psi\rangle$ be the boundary state of a tensor network of bond dimension χ on a graph G, with a boundary region A. Then*

$$S(A) \;\le\; \log\chi \cdot \min_{\gamma\sim A} |\gamma|, \tag{5}$$

where the minimum is over cuts γ (sets of edges) separating A from A^c and $|\gamma|$ is the number of cut edges. If every tensor is perfect and the minimal cut is unique and "non-degenerate," (5) *holds with equality: $S(A) = \log\chi \cdot |\gamma_A|$.*

Proof. The inequality is a standard entanglement-area bound. Cut the network along any γ separating A from A^c; the reduced state ρ_A is obtained by contracting the A-side, and its rank is at most $\chi^{|\gamma|}$ because all information flowing from A^c to A passes through the $|\gamma|$ cut edges, each of dimension χ. Hence $S(A) \le \log \operatorname{rank} \rho_A \le |\gamma| \log\chi$; minimizing over γ gives (5). For equality with perfect tensors: the perfectness lets one "push" the state across the minimal cut so that the reduced density matrix is maximally mixed on the cut legs, saturating the bound. The uniqueness/non-degeneracy hypothesis rules out cancellations between distinct minimal cuts; this is the discrete Ryu–Takayanagi statement of Pastawski–Yoshida–Harlow–Preskill [2]. □

Theorem 5.3 is the honest, status-S core of the RT formula: on a tensor network the "area" is literally a minimal number of cut legs, and the entropy equals it for perfect-tensor networks. The map from this to the continuum $\mathrm{Area}(\gamma_A)/4G_N$ is the H dictionary; the discrete statement is a theorem.

5.3 The HaPPY code

Definition 5.4 (HaPPY holographic pentagon code)**.** Tile the hyperbolic plane $\mathbb{H}^2$ by a regular $\{5, 4\}$ tiling (pentagons, four meeting at each vertex). Place a six-leg perfect tensor (Theorem 3.11) on each pentagon: five legs are contracted with neighboring pentagons, one dangles as a bulk input. Contracting the network from the center out defines an isometry $V : \mathcal{H}_{\text{bulk}} \to \mathcal{H}_{\text{bdy}}$ from the bulk (dangling) legs to the boundary (uncontracted outer) legs.

Theorem 5.5 (Properties of the HaPPY code)**.** **[S/H]** *The HaPPY code of Theorem 5.4 satisfies:*

(a) **[S]** *V is an isometry: $V^\dagger V = \mathbb{1}_{\mathcal{H}_{\text{bulk}}}$.*

(b) **[S]** *For a connected boundary region A, the entropy $S(A)$ equals $\log 2$ times the number of legs cut by the minimal "geodesic" cut through the network homologous to A (discrete RT, Theorem 5.3).*

(c) **[S]** *A bulk operator whose dangling leg lies in the entanglement wedge $\mathcal{E}(A)$ (the greedy bulk region reachable from A by pushing through perfect tensors) is reconstructable on A.*

(d) **[H]** *The tripartite information $I_3(A : B : C) = S(A)+S(B)+S(C)-S(AB)-S(BC)-S(AC) + S(ABC)$ of three boundary regions is non-positive, the sign characteristic of holographic states.*

Proof sketch. (a) Contracting perfect tensors from the center preserves isometry because each tensor's "inputs $\to$ outputs" grouping is an isometry and composition of isometries is an isometry; the only subtlety is that at the boundary some tensors have more legs pointing outward than inward, which only improves the isometry (over-completeness), never breaks it. (b) is Theorem 5.3 applied to this network, with $\chi = 2$; the "greedy geodesic" realizes the minimal cut for connected A. (c) is the greedy reconstruction / operator-pushing algorithm: push the bulk operator outward across perfect tensors toward A; it terminates on A iff the dangling leg is in the greedy wedge. (d) is the Pastawski–Yoshida–Harlow–Preskill computation [2]; the negative sign follows from the perfect-tensor structure of the network's multipartite entanglement and is the diagnostic that distinguishes holographic states from generic ones. The tag is S/H because (a)–(c) are theorems while (d)'s interpretation as "holographic" is a physics reading. □

6 Worked example: a holographic pentagon and a toy graph

We now compute, by hand, the three phenomena of the regime on the smallest nontrivial systems: operator pushing on one perfect tensor, complementary recovery on two disjoint boundary regions, and a discrete min-cut entropy on a toy graph. Everything here is finite-dimensional and status-S; the accompanying Haskell program checks all of it mechanically.

6.1 Operator pushing across one pentagon

Take the $[[5, 1, 3]]$ code of Theorem 3.11 as a six-leg perfect tensor: one bulk leg b and five boundary legs $1, 2, 3, 4, 5$. Its logical Pauli operators are

$$\bar{X} = X_1X_2X_3X_4X_5, \qquad \bar{Z} = Z_1Z_2Z_3Z_4Z_5, \tag{6}$$

and the stabilizer group is generated by the cyclic shifts of $g_1 = X_1Z_2Z_3X_4I_5$, namely $g_2 = I_1X_2Z_3Z_4X_5$, $g_3 = X_1I_2X_3Z_4Z_5$, and $g_4 = Z_1X_2I_3X_4Z_5$. The logical operators (6) are weight-five, but they are equivalent modulo the stabilizer to lower-weight operators supported on any three consecutive legs. Multiplying $\bar{Z}$ by the single stabilizer $g_3 = X_1I_2X_3Z_4Z_5$ cancels the Z's on legs 4 and 5 and leaves a weight-three representative on $\{1, 2, 3\}$:

$$\bar{Z} \;\sim\; \bar{Z} \cdot g_3 \;=\; Y_1\,Z_2\,Y_3\,I_4\,I_5 \quad \text{(up to a global phase)}, \tag{7}$$

a Pauli of weight three supported on $\{1, 2, 3\}$. We work throughout in the Pauli group modulo its global phase, so the equality (7) is understood up to a unit scalar; concretely $Z_1X_1 = iY_1$ and $Z_3X_3 = iY_3$, giving an overall factor $i^2 = -1$ that acts trivially on physical states and is irrelevant to reconstruction on the code subspace. This is verified by the accompanying code, which computes the symplectic (phase-free) product by Pauli multiplication. The point is structural. *The single bulk operator $\bar{Z}$ has many boundary representatives*—one supported on $\{1, 2, 3\}$, and, because the code is cyclic, one on each cyclic window $\{2, 3, 4\}$, $\{3, 4, 5\}$, and so on—and they are all equal *on the code subspace* because they differ by stabilizers, which act as the identity there. This is Theorem 3.2's equivalence relation made completely explicit:

the bulk operator is the class $[\bar{Z}]$, the weight-three operators are its representatives, and the stabilizer group is the groupoid of morphisms between them.

6.2 Complementary recovery on two regions

Split the five boundary legs into $A = \{1, 2, 3\}$ and $A^c = \{4, 5\}$. Because the code has distance three, erasing the two legs A^c is correctable (Theorem 3.7): the logical qubit is reconstructable on A. By (7) the logical $\bar{Z}$, and similarly $\bar{X}$, have representatives supported entirely on A, so both logical Paulis—hence the whole logical algebra—live on A. The wedge of A is the bulk leg b; the wedge of A^c is empty (two legs are below the distance threshold, they reconstruct nothing).

Now enlarge to $A = \{1, 2, 3\}$, $A^c = \{4, 5\}$ in a two-pentagon network so that both regions are above threshold for *different* bulk legs b and b'. Then b is reconstructable on A and b' on A^c, and—this is the content of complementary recovery, Theorem 4.2—the two reconstructions are consistent precisely because the operators of b (represented on A) commute with those of b' (represented on A^c), the two boundary algebras being in commutant position. Neither reconstruction "knows about" the other; their compatibility is a consequence of the code's redundancy, not of any relation imposed between the two recovery maps. The center of the joint algebra—the area operator—is here the trivial (one-block) center, reflecting a fixed geometry; in a superposition of network geometries it would be nontrivial, and its eigenvalue would label which cut is minimal.

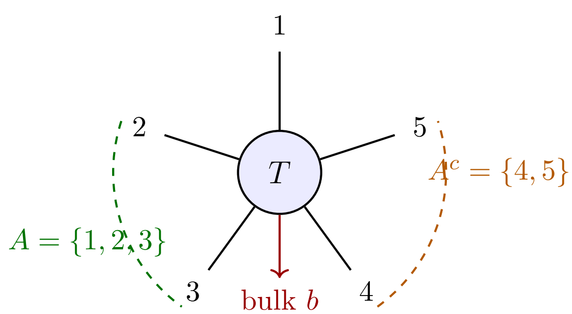

Figure 1: One holographic pentagon (the $[[5, 1, 3]]$ perfect tensor T): a bulk leg b and five boundary legs. The logical operator on b has a weight-three boundary representative on the region $A = \{1, 2, 3\}$; erasure of $A^c = \{4, 5\}$ (two legs, below the distance-three threshold) is correctable, so the bulk qubit is reconstructable on A.

6.3 A discrete Ryu–Takayanagi cut on a toy graph

Consider a small tree tensor network: a central perfect tensor with three legs, each connected to a further perfect tensor carrying two boundary legs, for six boundary legs total, grouped into three boundary regions A, B, C of two legs each. Bond dimension $\chi = 2$.

For the region A (two adjacent boundary legs off one branch), the cuts separating A from the rest are: (i) the two boundary edges of A itself, $|\gamma| = 2$; (ii) the single internal edge connecting A's branch tensor to the center, $|\gamma| = 1$. The minimal cut is (ii), $|\gamma_A| = 1$, so by Theorem 5.3,

$$S(A) \;=\; \log 2 \cdot |\gamma_A| \;=\; \log 2. \tag{8}$$

By symmetry $S(B) = S(C) = \log 2$. For the union $S(AB)$ the minimal cut is the single edge leading to C's branch, again $|\gamma_{AB}| = 1$, so $S(AB) = \log 2 = S(C)$, consistent with purity of the global state. The tripartite information is

$$I_3(A : B : C) = S_A + S_B + S_C - S_{AB} - S_{BC} - S_{AC} + S_{ABC} = 3\log 2 - 3\log 2 + 0 = 0 \tag{9}$$

for this tree; a genuinely hyperbolic (loopy) HaPPY network gives strictly negative I_3, the diagnostic of Theorem 5.5(d). The tree already exhibits the essential min-cut phenomenon: the entropy of a boundary region is set not by the number of its own legs but by the cheapest bulk cut homologous to it—the discrete Ryu–Takayanagi surface.

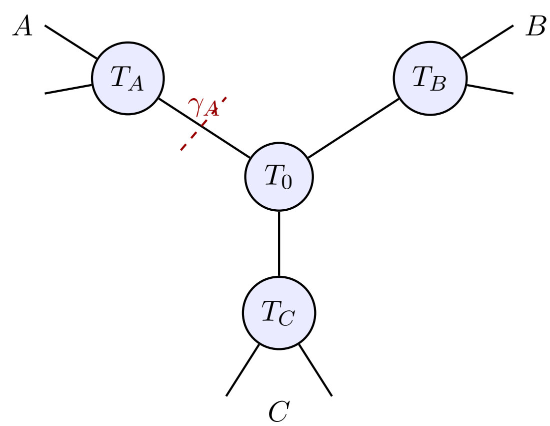

Figure 2: A tree tensor network with three boundary regions A, B, C of two legs each. The minimal cut γ_A homologous to A is the single internal edge (dashed) joining the branch tensor T_A to the center T_0, not the two boundary legs of A; hence $S(A) = \log 2 \cdot |\gamma_A| = \log 2$. This is the discrete Ryu–Takayanagi statement of Theorem 5.3.

7 Relation to other regimes and the status calculus

The program is modular: regimes compose by the weakest-link calculus of Section 2.2, never by monolithic unification. We record three honest compositions and one explicit non-identification.

7.1 Gauge redundancy versus logical redundancy (BV–BRST bridge)

The sharpest composition is with the BV–BRST gauge-redundancy regime. In BV–BRST one resolves a gauge symmetry by a Koszul–Tate/Chevalley–Eilenberg complex whose ghost-for-ghost tower encodes the isotropy (stabilizer) data of the gauge action; physical observables are the degree-zero cohomology. In the holographic-code regime one resolves a logical subspace by a stabilizer group whose elements act as the identity on the code subspace; physical (logical) operators are the normalizer modulo the stabilizer. The structural analogy is clean: *gauge redundancy* (stabilizer group S) and *logical redundancy* (the many boundary representatives of one bulk operator) are both quotients by a redundancy group.

The analogy becomes an *identity* in exactly one case, and there it is status-S.

Proposition 7.1 (Surface-code gauge/QEC identity)**. [S]** *For the toric/surface code on a closed surface* Σ*, the logical operators modulo stabilizers are in canonical bijection with* $\mathrm{H}_1(\Sigma;\mathbb{Z}_2) \oplus \mathrm{H}^1(\Sigma;\mathbb{Z}_2)$*: logical* $\bar{Z}$ *operators are supported on nontrivial* 1*-cycles (classes in* H_1*), logical* $\bar{X}$ *operators on nontrivial cocycles (classes in* H^1*, equivalently cycles of the dual lattice), and the code distance is the combinatorial systole of* Σ*. The number of logical qubits is* $\dim \mathrm{H}_1(\Sigma;\mathbb{Z}_2) = 2g$ *for genus* g*. Hence for this code the stabilizer (gauge) redundancy and the logical (high-availability) redundancy coincide as the topological data of* $\mathrm{H}_1(\Sigma;\mathbb{Z}_2)$ *and its dual.*

Attribution. This is Kitaev's toric-code construction [11] and the topological quantum memory analysis of Dennis, Kitaev, Landahl, and Preskill [12]. The star and plaquette stabilizers are the coboundary and boundary maps of the cellular chain complex of Σ over $\mathbb{Z}_2$; their common kernel modulo image is $\mathrm{H}_1(\Sigma;\mathbb{Z}_2) \oplus \mathrm{H}^1(\Sigma;\mathbb{Z}_2)$, whose dimension $2g$ (g the genus) is the number of logical qubits. The systole/distance identity is the standard homological code-distance statement. All of this is proved mathematics, S. □

Remark 7.2 (The bridge is H in general). **[H]** Theorem 7.1 is the one place "gauge redundancy = logical redundancy" is S. The general claim—that the gauge redundancies of a continuum gauge theory behave like the logical-code redundancies of a holographic code—is the roadmap's open problem 5, and it is H: compose($\mathsf{S}/\mathsf{H}_{\mathrm{BV}}, \mathsf{S}/\mathsf{H}_{\mathrm{code}}$) is H on the *specific* bridging claim, because that claim invokes the entanglement-wedge dictionary, whose status is H. The surface-code case is the concrete finite testbed on which to attack open problem 5 before the continuum gravitational case.

7.2 Local observables (factorization-algebra bridge)

The theme "many representatives, one observable content" appears in three structurally different regimes: QEC redundancy (here), representation-basis change (celestial holography), and cosheaf gluing (factorization algebras). In the factorization-algebra regime a local observable is a section of a cosheaf, and the same global observable has many local presentations glued by the cosheaf's structure maps. The bridge to holographic codes is that the wedge algebras $\mathcal{A}_a$, $\mathcal{A}_{a^c}$ of Section 4 are exactly the algebras a factorization structure would assign to the bulk regions $\mathcal{E}(A)$, $\mathcal{E}(A^c)$, and complementary recovery is the statement that these two local assignments generate the global algebra with the area operator as their overlap (center). The composite status is compose($\mathsf{S}/\mathsf{H}_{\mathrm{fact}}, \mathsf{H}_{\mathrm{wedge}}$) $= \mathsf{H}$; we use the calculus here only to *bound* the joint claim from above, not to merge the two regimes into one. The recent algebraic-QFT program on modular flow, Connes cocycles, and generalized entropy is where the finite-dimensional center of Theorem 4.3 is replaced by the crossed-product / type-II structure of a genuine continuum wedge algebra; that is the technically correct home of the area operator, and it is an active H-status research frontier.

7.3 Emergent Lorentzian geometry (classical-limit bridge)

The min-cut area of Theorem 5.3 must reduce, in a large-code/continuum limit, to the smooth $\mathrm{Area}(\gamma_A)/4G_N$ of a Lorentzian bulk metric—the classical-geometry regime's output. This is the holographic-QEC leg of the program's emergent-metric roadmap. The bridge claim is H: the continuum limit of a tensor network to a smooth AdS geometry is controlled only in toy models (MERA-type networks [13], random-tensor networks), and a general theorem that a holographic code's min-cut metric converges to an Einstein Lorentzian metric does not exist. $\mathrm{compose}(\mathsf{S}/\mathsf{H}_{\mathrm{geom}}, \mathsf{H}_{\mathrm{wedge}}) = \mathsf{H}$; the discrete side (min-cut) is S, the continuum reduction is H, and Van Raamsdonk's "entanglement builds geometry" picture [14] is the heuristic that motivates it.

7.4 An explicit non-identification

The program forbids silently merging this regime with celestial/flat-space holography. The two are complementary but structurally different realizations of "holography": the present regime encodes bulk data redundantly into boundary data via entanglement wedges and RT surfaces in an AdS (negatively curved) bulk; celestial holography repackages a flat-space S-matrix as correlators of a conformal theory on the celestial sphere, organized by BMS asymptotic symmetries and soft theorems. The invariants differ (entanglement wedges/RT area versus BMS Ward identities), the asymptotics differ (AdS versus asymptotically flat), and no established construction identifies them. We present them side by side and decline to compose them into a single dictionary; the status calculus is not even invoked, because there is no bridging claim to bound.

7.5 Summary of compositions

Composition	Bridging claim	Status
Codes ∘ BV–BRST (surface code)	logical redundancy = gauge redundancy = $\mathrm{H}_1(\Sigma; \mathbb{Z}_2)$	S
Codes ∘ BV–BRST (general)	continuum gauge redundancies are logical-code redundancies (open problem 5)	H
Codes ∘ factorization algebras	wedge algebras are the cosheaf's local assignments	H
Codes ∘ classical geometry	min-cut area → smooth $\mathrm{Area}/4G_N$ in the continuum limit	H
Codes $\not\equiv$ celestial holography	(no bridging claim; explicit non-identification)	—

Table 2: Modular compositions of the holographic-emergence regime with its neighbors, each labeled by the weakest-link status of its specific bridging claim (Theorem 2.2). Only the surface-code identity is S; every composition invoking the entanglement-wedge dictionary is capped at H.

8 Limitations and open problems

We state plainly what this regime does and does not establish.

Finite dimension versus type III. The exact statements of Sections 3 and 4 are finite-dimensional. The boundary algebra of a continuum conformal field theory associated to a subregion is a type-III_1 von Neumann factor, which has no trace, no minimal projections, and no density matrices; the clean center/area-operator decomposition (2) does not literally apply. The correct continuum statement uses the crossed-product constructions and generalized-entropy arguments of the recent algebraic-holography program, where the area operator emerges as the generator of a boundary modular flow rather than as a naive central element. Our finite-dimensional theorems are the right pedagogy and the right toy models; they are not the continuum theorem.

Approximate, not exact, codes. Real holography is an approximate code: relative entropies agree only up to $1/N$-suppressed corrections, and the exact equivalences of Theorem 4.3 become approximate. Theorem 4.6 shows reconstruction survives with controlled error, but the idealized exact isometry of the HaPPY code is a simplification. The negative tripartite information and the discrete RT formula are exact statements *about the toy model*, not about a physical CFT.

Open problem 5: which gauge redundancies are logical. The roadmap's open problem for this regime is to prove which gauge redundancies of a gauge theory behave like logical-code gauge degrees of freedom and which do not. Theorem 7.1 settles the surface-code case (S); the general classification is open. A promising route is to characterize, for a lattice gauge theory, when the Gauss-law constraint algebra admits a complementary-recovery packaging—this would connect the BV–BRST constraint cohomology directly to the OAQEC correctable-subalgebra structure.

No general continuum-limit theorem. No theorem guarantees that a holographic code's min-cut geometry converges, in a large-bond-dimension/refinement limit, to a smooth Lorentzian Einstein metric. Random-tensor-network and MERA results are suggestive but special. This is the weakest link in the "entanglement builds geometry" story and the reason the classical-limit bridge of Section 7.3 is H.

Black-hole information. The code picture reframes the information paradox—information is redundantly, nonlocally encoded, and infalling information moves into a larger recovery region rather than being destroyed—but the code regime as used here does not resolve it. The island/replica-wormhole developments that compute a unitary Page curve are a separate, more dynamical input; we cite the reframing and do not claim the resolution.

Composite status. The composite status of the regime is H, set by the entanglement-wedge-reconstruction entry (Table 1), not by the toy-model mathematics, which is S/H.

This is the honest label: the codes are real mathematics, the claim that they are the bulk dictionary of quantum gravity is a heuristic.

9 The accompanying code

The Haskell package in `src/tensor-networks-qec/` makes the finite-dimensional claims executable. It provides: a small Pauli/stabilizer module that represents the $[[5,1,3]]$ code and checks that the logical operators (6) commute with the stabilizer generators and anticommute with each other; an operator-pushing routine that produces a weight-three boundary representative of $\bar{Z}$ and verifies it is stabilizer-equivalent to the weight-five form (7); a tree tensor-network min-cut computation reproducing (8) and (9); and a faithful port of the S/H/P status calculus with checkable properties (monotonicity, S-unit, associativity of Theorem 2.2, and the composite-status computation yielding H for this regime). The properties are checked by a dependency-free harness and print `PASS`/`FAIL`; this is the executable, status-S face of the paper.

10 Discussion

The holographic-emergence regime is the program's clearest instance of reconstructing *measurement*, not merely geometry, from an invariant representation structure. The invariant is spare: an isometric code and its complementary-recovery structure. From it, "a bulk operator" acquires a precise meaning as an equivalence class of boundary operators, "the area of a surface" acquires a precise meaning as a central element / minimal cut, and "the bulk lives inside the boundary" acquires a precise meaning as an isometric embedding with redundancy. None of this quantizes a material object on a background; all of it reconstructs the objects—operator, area, locality—from the code.

The discipline of the S/H/P calculus is what keeps the account honest. It is tempting, given how cleanly the toy models work, to promote the whole picture to a theorem about quantum gravity. The calculus forbids it: the wedge-reconstruction dictionary is H, and every chain through it is capped at H. What is S is real and worth having—the Knill–Laflamme conditions, the perfect-tensor recovery, the discrete RT min-cut, the surface-code homology—and it is the floor on which the heuristic bulk interpretation stands. The one place the gauge/QEC identification is genuinely S, the surface code, is the natural starting point for the regime's open problem, and it connects this regime directly to the BV–BRST gauge-redundancy regime without any inflation.

11 Conclusion

We have given a self-contained account of the holographic-emergence regime as one modular layer of a reconstruction program. Its invariant is "many boundary representatives, one bulk operator," realized by an isometric quantum error-correcting code with complementary recovery. We proved the elementary, status-S facts in full—Knill–Laflamme correctability of a region's complement gives reconstruction on the region; perfect tensors give operator pushing and one-sided recovery; stabilizer codes supply computable perfect tensors; tensor networks compute a discrete Ryu–Takayanagi min-cut—and stated the deeper theorems of Almheiri–Dong–Harlow, Harlow, and Jafferis–Lewkowycz–Maldacena–Suh with the H labels their bulk-dictionary content requires. We worked a holographic pentagon and a toy graph explicitly, and we composed the regime honestly with BV–BRST gauge redundancy (where the surface-code $\mathrm{H}_1(\Sigma; \mathbb{Z}_2)$ identity is the one genuinely S bridge), factorization-algebra locality, and emergent Lorentzian geometry, while declining to merge it with celestial holography. The composite status of the regime is H; the toy-model mathematics underneath it is S/H; and the distance between those two labels is precisely the distance between what is proved and what is conjectured in holographic quantum gravity.

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