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Quantum Spatial Geometry as Representation Data: Spin Networks, Intertwiners, and the Geometric Operators of the Boundary Hilbert Space

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Abstract

We develop quantum spatial geometry as representation-theoretic data attached to a graph, following the reconstruction thesis that quantum gravity is not the quantization of matter placed in a fixed spacetime but the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures. Concretely, a state of quantum three-geometry is a spin network: a graph Γ carrying an irreducible SU(2) representation j_e on each edge and an intertwiner ι_v at each vertex, taken modulo spatial diffeomorphisms. We give a self-contained account of the Ashtekar–Isham–Lewandowski kinematical Hilbert space $\mathcal{H}_{\mathrm{kin}} = L^2(\overline{\mathcal{A}}, \mu_{\mathrm{AL}})$, review the proof (due to Baez and to Ashtekar–Lewandowski) that the spin-network states form an orthonormal basis through the Peter–Weyl theorem, and show that imposing the Gauss constraint at each vertex reduces the local Hilbert space to the intertwiner space $\mathrm{Inv}\big(\bigotimes_{e\ni v} V_{j_e}\big)$. Geometry enters through operators: we reconstruct the Rovelli–Smolin–Ashtekar–Lewandowski area operator $\hat{A}(S)$ and its proof of essential self-adjointness, with pure point spectrum $8\pi\gamma\ell_{\mathrm{P}}^2 \sum_i \sqrt{j_i(j_i+1)}$ on transversal punctures and a transversal area gap $4\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2$; we describe the volume operator, its support on vertices of valence at least four, and the inequivalence of the Rovelli–Smolin and Ashtekar–Lewandowski regularizations together with the flux-consistency criterion that selects the latter. Three worked examples—the theta graph, the four-valent spin-$\frac{1}{2}$ vertex, and a surface pierced by n fundamental edges—make the representation-theoretic content explicit. We recall the Lewandowski–Okołów–Sahlmann–Thiemann uniqueness theorem, which fixes this representation among all diffeomorphism-invariant ones. Throughout we tag every claim with the project's S/H/P epistemic calculus: the kinematical results are standard mathematics ([S]), while the physical identification of these structures with the geometry of our universe and the recovery of a smooth three-metric in a large-spin limit remain heuristic ([H]) and are flagged as open. We close by locating spin networks among the eleven other regimes

of the representation library—in particular as the boundary data on which spin-foam dynamics acts and as a graph-local instance of the "physical content is a groupoid, not a coarse quotient" principle—and by stating the continuum-limit problem honestly rather than as resolved.

Contents

1 Introduction

1.1 The reconstruction thesis

There is a habitual way to state the problem of quantum gravity: take the gravitational field, a metric $g_{\mu\nu}$ on a spacetime manifold, and quantize it the way one quantizes any other field. On this reading gravity is one more material object sitting inside spacetime, and quantization is a procedure applied to it. The trouble is that the spacetime is exactly what the metric is supposed to be. General covariance says that the manifold coordinates carry no physics; the physics is in the equivalence class of the metric under diffeomorphisms, together with the residual symmetry (Killing) data that a coarse quotient would throw away. Once one takes that seriously, the "field to be quantized" and the "arena it lives in" stop being separable, and the naive picture loses its footing.

This paper adopts a different organizing principle, shared across a modular library of twelve quantum-gravity regimes:

> *Quantum gravity is not primarily the quantization of material objects in spacetime. It is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures.*

In the regime treated here—quantum spatial geometry—the invariant representation structure is explicit and finite-dimensional edge by edge. A state of quantum three-geometry is not a smoothed-out Riemannian metric on a Cauchy slice. It is a *spin network*: a graph Γ embedded in a spatial three-manifold Σ, with an irreducible representation of SU(2) assigned to each edge and an invariant tensor (intertwiner) assigned to each vertex, all taken modulo the spatial diffeomorphisms of Σ. The reconstruction thesis is not a slogan here but a computation: area and volume are *defined* as the spectra of operators built from these representation labels, and a smooth three-geometry is recovered—if at all—only as a coarse-grained limit of the underlying combinatorial-representation-theoretic data. Penrose anticipated exactly this inversion in 1971, proposing that "combinatorial spacetime" built from angular-momentum (SU(2)) recoupling might precede, rather than presuppose, the continuum [1].

1.2 What is settled and what is not

Loop quantum gravity supplies the rigorous core. The kinematical Hilbert space $\mathcal{H}_{\text{kin}}$ of the theory, built by Ashtekar, Isham, and Lewandowski from cylindrical functions of an SU(2) connection, is a genuine, well-defined object of functional analysis [4, 3]. The spin-network states form an orthonormal basis of it. The area and volume operators are honest self-adjoint (or symmetric) operators with discrete spectra [2, 5, 6]. The representation itself is unique among diffeomorphism-invariant representations of the holonomy–flux algebra, by the theorem of Lewandowski, Okołów, Sahlmann, and Thiemann [7], with an independent Weyl-algebra version due to Fleischhack [8]. These are theorems, and we treat them as such.

What is *not* settled is the physics that would turn this kinematics into a theory of nature: the dynamics (the Hamiltonian constraint, or covariantly the spin-foam sum), and above all the continuum/semiclassical limit in which a smooth Einsteinian three-geometry should re-emerge. We are careful to keep these apart. The paper uses the library's three-valued epistemic calculus, with warrants ordered $\mathsf{S} < \mathsf{H} < \mathsf{P}$:

- [S] a standard mathematical or mathematical-physics correspondence (a theorem in a well-posed setting);
- [H] a strong heuristic physical representation;
- [P] a speculative ontological extension.

Composite claims inherit the *worst* warrant of their inputs; speculative entries are never silently upgraded. Every displayed result below carries a tag. The reader should come away knowing precisely which statements are proved and which are physical hypotheses.

1.3 Contributions and outline

This is primarily an expository reconstruction, but it is organized around a specific claim—that the geometry *is* the representation data—and it makes that claim checkable. Our contributions are:

1. A clean statement of the representation-stack/realization pipeline specialized to spin networks (Section 2), with the S/H/P tags made explicit.
2. A self-contained construction of $\mathcal{H}_{\text{kin}}$, the spin-network basis, and the intertwiner reduction at nodes (Sections 4 and 6), including the character-integral dimension formula and a worked four-valent count.

3. A self-contained reconstruction of the proof that the area operator is essentially self-adjoint with the stated pure point spectrum and area gap (Theorem 7.2, following Rovelli–Smolin and Ashtekar–Lewandowski), and a careful account of the volume operator including the Rovelli–Smolin versus Ashtekar–Lewandowski inequivalence and the Giesel–Thiemann consistency criterion (Section 8).

4. Three worked examples (Section 9) that expose the recoupling theory concretely.

5. A statement of the LOST uniqueness theorem and its role (Section 11), and an honest treatment of the continuum-limit problem (Section 14).

6. A map of how this regime composes with the other eleven in the library, using the weakest-link status calculus (Section 12), together with an accompanying Haskell model whose intertwiner counts and area eigenvalues are machine-checkable (Section 13).

Notation: Σ is a smooth, oriented, analytic three-manifold (the spatial slice); $\ell_{\mathrm{P}}^2 = \hbar G/c^3$ is the Planck area; $\gamma > 0$ is the Barbero–Immirzi parameter; G is Newton's constant. We set $c = 1$ but keep $\hbar$, G, γ visible where they carry physics. Representations of SU(2) are labelled by spins $j \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, with carrier space V_j of dimension $2j+1$ and Casimir eigenvalue $j(j+1)$.

2 Mathematical framework: reconstruction from representation data

We first fix the abstract skeleton the whole library shares, then instantiate it.

2.1 The realization pipeline

A *representation entry* is a tuple $E = (M, P, \tau, \sigma)$: a mathematical object M, a physical target P, a translation datum τ carrying assumptions and limitations, and an epistemic status $\sigma \in \{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$. Observables are produced by a realization pipeline

$$\begin{gathered} M \;\longrightarrow\; \Phi(M) \;\longrightarrow\; \mathrm{Real}_\alpha(\Phi(M)) \;\longrightarrow\; \text{physical representation} \;\longrightarrow\; \mathrm{Obs}, \\ \mathrm{Obs}_\alpha(M) = \mathrm{Obs}\big(\mathrm{Real}_\alpha(\Phi(M))\big). \end{gathered} \tag{1}$$

For spin networks the pipeline reads: (graph with SU(2) labels) $\to$ (cylindrical function on generalized connections) $\to$ (vector in $\mathcal{H}_{\mathrm{kin}}$) $\to$ (geometric operator) $\to$ (area/volume eigenvalue). The point of the framing is that the last arrow, "geometric operator $\to$ eigenvalue," is where geometry is *produced*, not read off.

2.2 The status calculus

Definition 2.1 (Warrant composition, [S])**.** Let the warrants be ordered $\mathsf{S} < \mathsf{H} < \mathsf{P}$. For a finite family of warrants $\sigma_1, \ldots, \sigma_n$, define the composite

$$\sigma_1 * \cdots * \sigma_n \;=\; \max_{1 \leq i \leq n} \sigma_i. \tag{2}$$

Then S is a two-sided identity ($\mathsf{S} * \sigma = \sigma$), $*$ is associative and commutative, and it is monotone: $\sigma \leq \sigma'$ implies $\sigma * \rho \leq \sigma' * \rho$.

The content is a discipline: a chain of reasoning that passes through a heuristic step is at best heuristic, no matter how rigorous its other steps. We will invoke this repeatedly. For example, the kinematical theorems below are [S]; the statement "this is the quantum geometry of the physical world" composes them with an [H] physical-target identification and is therefore [H].

2.3 The relevant dictionary entries

The library records each mathematical object together with its physical target, the translation datum, and a status. Table 1 reproduces the entries that bear directly on this paper. The row for spin networks proper carries [S/H]: its internal kinematical theorems are [S], but the physical identification with the geometry of our universe is [H], and the composite is the worse of the two. The adjacent rows (gauge groupoid, connection variables, spin foams) fix the neighbours we compose with in Section 12.

Math object	**Physical representation**	**Translation / caveat**	**Status**
Principal SU(2) bundle with connection A_a^i and triad E_i^a	gravitational gauge/geometry variables on a slice	Ashtekar–Barbero phase space; variable choice differs across formulations	[S/H]
Gauge groupoid / quotient stack	many representatives, one observable content	$\mathrm{Aut}_{[X/G]}(x) \simeq \mathrm{Stab}_G(x)$; isotropy retained, not erased	[S]
Spin network: graph labelled by SU(2) reps	quantum geometry boundary state	spin-network realization; semiclassical extraction remains nontrivial	[S/H]
Area / volume functional	discrete geometric observable	spectra from SU(2) Casimir/recoupling; scale set by γ	[S/H]
Spin-foam 2-complex with face/edge labels	covariant transition amplitude (the dynamics)	state-sum realization; continuum/refinement limit open	[H]

Table 1: The dictionary entries used in this paper, with their S/H/P status. The composite status of a claim built from several rows is the worst warrant among them (Theorem 2.1).

2.4 Gauge as retained isotropy, not discarded redundancy

A recurring structural fact threads the library. When a group G acts on a space X, the physically correct object is the action groupoid (quotient stack) $[X/G]$, whose automorphism group at a point x is the stabilizer,

$$\mathrm{Aut}_{[X/G]}(x) \;\simeq\; \mathrm{Stab}_G(x), \tag{3}$$

rather than the coarse orbit set X/G, which forgets stabilizers. In the classical gravitational regime this is $\mathrm{Aut}(g) \simeq \mathrm{Isom}(M, g)$, the isometry group of a metric [23]. We will see the graph-local avatar of the same fact in Section 6: imposing SU(2) gauge invariance at a vertex does not delete the vertex degrees of freedom, it replaces them by the intertwiner space, which is exactly the "retained isotropy" at that node. Geometry lives on this retained data.

3 The Ashtekar–Barbero phase space and holonomy–flux algebra

3.1 Connection variables

Canonical loop quantum gravity starts from a Hamiltonian formulation of general relativity in connection variables. On a spatial slice Σ one has an $\mathfrak{su}(2)$ (Ashtekar–Barbero) connection A_a^i and a densitized triad E_i^a, with a a spatial tensor index and i an internal $\mathfrak{su}(2)$ index, related to the ADM data by $E_i^a E_i^b = \det(q)\, q^{ab}$ for the spatial metric q_{ab} and $A_a^i = \Gamma_a^i + \gamma K_a^i$, where Γ is the spin connection of the triad, K the extrinsic curvature, and γ the Barbero–Immirzi parameter [9, 10]. The canonical bracket is

$$\{A_a^i(x), E_j^b(y)\} = 8\pi G\gamma\, \delta_a^b\, \delta_j^i\, \delta^3(x, y). \tag{4}$$

The theory is a constrained system: Gauss, spatial-diffeomorphism, and Hamiltonian constraints generate, respectively, internal SU(2) rotations, spatial diffeomorphisms of Σ, and (in the covariant reading) refoliations.

3.2 Holonomies and fluxes

Point-valued fields are too singular to quantize directly in a background-independent way. One smears them:

Definition 3.1 (Holonomy, [S])**.** For an oriented piecewise-analytic edge $e \subset \Sigma$ and a connection A, the holonomy is the path-ordered exponential

$$h_e[A] = \mathrm{Hol}_e(A) = \mathcal{P}\exp\int_e A \;\in\; \mathrm{SU}(2). \tag{5}$$

Under a gauge transformation $g : \Sigma \to \mathrm{SU}(2)$, $h_e \mapsto g(s(e))\, h_e\, g(t(e))^{-1}$, where $s(e), t(e)$ are the source and target of e.

Definition 3.2 (Flux, [S])**.** For a two-surface $S \subset \Sigma$ and an $\mathfrak{su}(2)$-valued smearing f^i, the flux is

$$E(S, f) = \int_S \epsilon_{abc}\, E_i^a f^i\, dx^b \wedge dx^c. \tag{6}$$

Holonomies and fluxes close into a Poisson $*$-algebra $\mathfrak{a}$, the *holonomy–flux algebra*, whose brackets are graph-local: $E(S, f)$ acts on h_e only through the intersection points $S \cap e$, essentially by inserting an $\mathfrak{su}(2)$ generator at each transversal intersection. This graph-locality is the technical seed of the discreteness of geometry.

4 Cylindrical functions and the kinematical Hilbert space

4.1 Cylindrical functions

Definition 4.1 (Cylindrical function, [S]). Let $\Gamma = \{e_1, \dots, e_n\}$ be a finite embedded graph in Σ. A function Ψ of the connection is *cylindrical over* Γ if there is a continuous $f : \mathrm{SU}(2)^n \to \mathbb{C}$ with

$$\Psi_{\Gamma,f}[A] = f\big(h_{e_1}[A], \dots, h_{e_n}[A]\big). \tag{7}$$

Write Cyl_Γ for these and $\mathrm{Cyl} = \bigcup_\Gamma \mathrm{Cyl}_\Gamma$ for all cylindrical functions (over all finite graphs).

Cyl is an abelian $*$-algebra (a graph refining two graphs carries both). Its Gelfand spectrum is the space $\overline{\mathcal{A}}$ of *generalized connections*: assignments $e \mapsto h_e \in \mathrm{SU}(2)$ to edges, compatible with composition and inversion of paths, not required to arise from any smooth A. Equivalently, $\overline{\mathcal{A}} = \varprojlim_\Gamma \mathrm{SU}(2)^{|\Gamma|}$ is a projective limit of copies of the compact group, hence itself a compact Hausdorff space [4].

4.2 The Ashtekar–Lewandowski measure

Theorem 4.2 (AL measure and kinematical Hilbert space, [S]). *There is a unique regular Borel probability measure μ_{AL} on $\overline{\mathcal{A}}$ such that, for every graph Γ with n edges and every $f \in C(\mathrm{SU}(2)^n)$,*

$$\int_{\overline{\mathcal{A}}} \Psi_{\Gamma,f}\, d\mu_{\mathrm{AL}} = \int_{\mathrm{SU}(2)^n} f(g_1, \dots, g_n)\, dg_1 \cdots dg_n, \tag{8}$$

with dg the normalized Haar measure. The completion

$$\mathcal{H}_{\mathrm{kin}} = L^2(\overline{\mathcal{A}}, \mu_{\mathrm{AL}}) \tag{9}$$

is a separable Hilbert space, and Cyl *is dense in it.*

Proof sketch. Consistency of the family of Haar integrals under graph refinement (subdividing an edge, or adding an edge with the trivial representation) is exactly the invariance and normalization of Haar measure; the Kolmogorov/Prokhorov consistency theorem for projective limits of compact groups then yields a unique cylindrical measure that extends to a regular Borel measure on the compact space $\overline{\mathcal{A}}$. Separability follows because the piecewise-analytic graphs form a countable directed set up to the relevant equivalence, and each $L^2(\mathrm{SU}(2)^n)$ is separable. See [4, 11, 12] for details. □

The measure μ_{AL} is diffeomorphism invariant and gauge invariant by construction: Haar measure does not see the embedding, only the combinatorics and the group. This invariance is what makes $\mathcal{H}_{\mathrm{kin}}$ the right home for a background-independent theory, and it is precisely the hypothesis of the uniqueness theorem in Section 11.

5 The spin-network basis

5.1 Peter–Weyl on each edge

The Peter–Weyl theorem decomposes $L^2(\mathrm{SU}(2))$ into matrix coefficients of irreducibles:

$$L^2(\mathrm{SU}(2)) \;\cong \bigoplus_{j\in\frac{1}{2}\mathbb{Z}_{\geq 0}} V_j \otimes V_j^*, \qquad \{\sqrt{2j+1}\,[\pi_j(g)]_{mn}\}_{j,m,n} \text{ orthonormal.} \tag{10}$$

Applying this edge by edge to $L^2(\mathrm{SU}(2)^n)$ and organizing the result gives a basis of $\mathcal{H}_{\mathrm{kin}}$ labelled by a graph, a spin per edge, and a pair of magnetic indices per edge; contracting the magnetic indices at each vertex against a basis of invariant tensors packages the data into spin networks.

Definition 5.1 (Spin-network state, [S])**.** A *spin network* is a triple $s = (\Gamma, \{j_e\}, \{\iota_v\})$: a finite oriented graph $\Gamma \subset \Sigma$, a spin $j_e \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ on each edge e, and, at each vertex v, an intertwiner

$$\iota_v \in \mathrm{Inv}\Big(\bigotimes_{e:\,t(e)=v} V_{j_e} \;\otimes \bigotimes_{e:\,s(e)=v} V_{j_e}^*\Big), \tag{11}$$

i.e. an SU(2)-invariant vector in the tensor product of the incoming and dual outgoing edge representations at v. The associated state $|s\rangle \in \mathcal{H}_{\mathrm{kin}}$ is obtained by evaluating the representation matrices $\pi_{j_e}(h_e)$ and contracting all magnetic indices with the intertwiners.

Theorem 5.2 (Spin-network basis, [S])**.** *The spin-network states $\{|s\rangle\}$, ranging over all graphs, edge spins, and orthonormal bases of the intertwiner spaces, form an orthonormal basis of the gauge-invariant kinematical Hilbert space $\mathcal{H}^G_{\mathrm{kin}}$. Before imposing gauge invariance, the analogous states—formed by contracting the edge matrices $\pi_{j_e}(h_e)$ against a complete orthonormal basis of the full (unprojected) tensor product $\bigotimes_{e\ni v} V_{j_e}$ at each vertex, rather than against the invariant subspace—give an orthonormal basis of $\mathcal{H}_{\mathrm{kin}}$ itself.*

Proof sketch. Orthonormality: distinct graphs can be refined to a common graph, on which two spin-network functions are products of Wigner matrix coefficients in distinct irreducibles or with distinct contraction patterns; Peter–Weyl orthogonality and Haar-measure normalization from Theorem 4.2 give $\langle s \mid s'\rangle = \delta_{ss'}$ once intertwiner bases are chosen orthonormal. Completeness: Peter–Weyl gives density of the span of matrix coefficients in each $L^2(\mathrm{SU}(2)^n)$; projecting onto the gauge-invariant subspace by averaging over SU(2) at each vertex sends matrix coefficients to intertwiner contractions, and the image spans $\mathcal{H}^G_{\mathrm{kin}}$. See [3, 5] for the full argument. □

This is the sense in which "the representation labels are the geometry": the Hilbert space of quantum three-geometry has a distinguished orthonormal basis whose labels are nothing but SU(2) representations on a graph and invariant tensors at its nodes. There is no metric anywhere in Theorem 5.1.

6 Gauge invariance at nodes: intertwiners

6.1 The Gauss constraint as a projection

The Gauss constraint generates internal SU(2) rotations. Its imposition is a projection onto gauge-invariant states, which acts vertex by vertex.

Proposition 6.1 (Gauss quotient equals the intertwiner space, [S])**.** *Let v be a vertex of valence m with adjacent edge spins $j_1, \dots, j_m$ (dualize those oriented outward). The gauge-invariant subspace of the local vertex Hilbert space is*

$$\mathcal{K}_v = \mathrm{Inv}\Big(\bigotimes_{i=1}^{m} V_{j_i}\Big) = \mathrm{Hom}_{\mathrm{SU}(2)}\Big(\mathbb{C},\ \bigotimes_{i=1}^{m} V_{j_i}\Big), \tag{12}$$

the space of SU(2)*-invariant tensors. Its dimension is*

$$\dim \mathcal{K}_v = \frac{2}{\pi}\int_0^{\pi} \Big(\prod_{i=1}^{m} \frac{\sin\big((2j_i+1)\theta\big)}{\sin\theta}\Big) \sin^2\theta \, d\theta, \tag{13}$$

the multiplicity of the trivial representation in $\bigotimes_i V_{j_i}$.

Proof. Gauge transformations act on the vertex by the diagonal SU(2); the invariant vectors are the multiplicity space of the trivial representation in the tensor product. By Schur orthogonality of SU(2) characters $\chi_j(\theta) = \sin((2j+1)\theta)/\sin\theta$ with respect to the Weyl measure $\frac{2}{\pi}\sin^2\theta\, d\theta$ on the maximal torus,

$$\dim \mathrm{Inv}\Big(\bigotimes_i V_{j_i}\Big) = \int_{\mathrm{SU}(2)} \prod_i \chi_{j_i}(g)\, dg = \frac{2}{\pi}\int_0^{\pi} \prod_i \chi_{j_i}(\theta)\, \sin^2\theta\, d\theta, \tag{14}$$

which is eq. (13). □

This is the graph-local instance of the "isotropy retained" principle of Section 2: the gauge quotient at v is not empty and not a mere point; it is the finite-dimensional representation-theoretic space $\mathcal{K}_v$, and the volume operator will act precisely on it.

6.2 Recoupling and the four-valent case

For a trivalent vertex, $\dim \mathcal{K}_v \in \{0, 1\}$: the invariant is unique when it exists, and it exists iff the three spins satisfy the triangle inequalities $|j_1 - j_2| \le j_3 \le j_1 + j_2$ and the integrality condition $j_1 + j_2 + j_3 \in \mathbb{Z}$. The unique invariant is the coordinate-free three-valent intertwiner tensor, whose components in the standard magnetic basis are given by the Wigner $3jm$-symbol $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ (equivalently, up to normalization and a metric factor, the Clebsch–Gordan coefficients).

For a four-valent vertex the intertwiner space is generically higher-dimensional, and a basis is given by recoupling: fixing a pairing, say (12)(34), an intermediate spin k that appears in both $V_{j_1} \otimes V_{j_2}$ and $V_{j_3}^* \otimes V_{j_4}^*$ labels a basis vector, and

$$\dim \mathcal{K}_v = \#\{k :\ \max(|j_1 - j_2|, |j_3 - j_4|) \le k \le \min(j_1 + j_2,\, j_3 + j_4)\}. \tag{15}$$

Different pairings give different bases related by the Wigner $6j$-symbols. This recoupling combinatorics is the entire content of the vertex Hilbert space; the volume operator, below, is a concrete matrix on this finite basis.

7 Geometric operators I: area

7.1 Construction

The classical area of a surface S is $\mathrm{Area}(S) = \int_S \sqrt{E^a_i E^b_i\, n_a n_b}\, d^2\sigma$ with n_a the conormal. Regularizing S by a partition into cells, promoting fluxes to operators, and taking the limit yields a well-defined operator on $\mathcal{H}_{\mathrm{kin}}$ [2, 5]. The single fact that drives the whole computation is how a flux operator acts on a holonomy, which we isolate.

Lemma 7.1 (Flux acts as an invariant vector field, [S])**.** *Let e be an edge crossing a small surface cell S_I transversally at one interior point, and let $\hat{E}^i(S_I)$ be the corresponding (self-adjoint) flux operator. Fix the $\mathfrak{su}(2)$ generators to be anti-Hermitian, $\tau^i = -\frac{i}{2}\sigma^i$ in the fundamental, so that $\pi_j(\tau^i)\pi_j(\tau_i) = -C_2(j)\,\mathbf{1}$ with $C_2(j) = j(j+1)$ the* SU(2) *Casimir on the irreducible V_j. On the holonomy h_e carried in the spin-j representation, $\hat{E}^i(S_I)$ acts by inserting the Hermitian combination $-i\,\pi_j(\tau^i)$ at the intersection point:*

$$\hat{E}^i(S_I)\,\pi_j(h_e) = -i\,8\pi\gamma\ell_{\mathrm{P}}^2\;\pi_j(h_{e'})\,\pi_j(\tau^i)\,\pi_j(h_{e''}), \tag{16}$$

where $e = e' \circ e''$ is split at the intersection. The factor $-i$ makes $\hat{E}^i(S_I)$ self-adjoint ($-i\,\tau^i = -\frac{1}{2}\sigma^i$ is Hermitian). Consequently, on a single transversal edge,

$$\begin{aligned}\sum_i \hat{E}^i(S_I)\hat{E}^i(S_I) &= (8\pi\gamma\ell_{\mathrm{P}}^2)^2\,(-i)^2\;\pi_j(\tau^i)\pi_j(\tau_i)\\ &= -(8\pi\gamma\ell_{\mathrm{P}}^2)^2\,\pi_j(\tau^i)\pi_j(\tau_i) = (8\pi\gamma\ell_{\mathrm{P}}^2)^2\,j(j+1)\,\mathbf{1},\end{aligned} \tag{17}$$

where the two sign flips—from $(-i)^2 = -1$ and from $\pi_j(\tau^i)\pi_j(\tau_i) = -j(j+1)\mathbf{1}$—cancel to give a strictly positive operator.

Proof. The flux $E(S_I, f) = \int_{S_I} \epsilon_{abc} E^a_i f^i$ has, with the connection, the Poisson bracket of a left/right-invariant vector field on the copy of SU(2) carried by e; upon canonical quantization the momentum flux becomes $\hat{E}^i(S_I) = -i\,8\pi\gamma\ell_{\mathrm{P}}^2\,X^i_e$, with X^i_e the right-invariant vector field generating $\pi_j(\tau^i)$ insertion, the $-i$ being the usual factor rendering a momentum operator self-adjoint. The intersection number of e with S_I being one, only the single insertion survives. Squaring and summing over i gives (minus) the quadratic Casimir times $(-i)^2$, i.e. the positive scalar $j(j+1)$ on the irreducible by Schur's lemma. See [5, 12]. □

Theorem 7.2 (Area operator: self-adjointness and spectrum, [S])**.** *Let $S \subset \Sigma$ be an analytic two-surface. There is an operator $\hat{A}(S)$ on $\mathcal{H}_{\mathrm{kin}}$, essentially self-adjoint on* Cyl*, that is diagonal in the spin-network basis. If a spin-network state $|s\rangle$ has edges $e_1, \ldots, e_N$ crossing S transversally with spins $j_1, \ldots, j_N$ and no edge lying in S or ending on S, then*

$$\hat{A}(S)\,|s\rangle = 8\pi\gamma\ell_{\mathrm{P}}^2 \sum_{i=1}^{N} \sqrt{j_i(j_i+1)}\;|s\rangle. \tag{18}$$

The spectrum is pure point (no continuous part), and the smallest nonzero eigenvalue in the transversal sector (the transversal area gap*, one transversal spin-$\frac{1}{2}$ edge) is*

$$\Delta A_\perp = 8\pi\gamma\ell_{\mathrm{P}}^2\sqrt{\tfrac{1}{2}\cdot\tfrac{3}{2}} = 4\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2. \tag{19}$$

Remark 7.3 (Transversal versus absolute area gap, [S]). The value $\Delta A_\perp = 4\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2$ is the gap in the transversal sector only. If one allows nodes to lie *on* S, the refined formula eq. (20) below admits smaller eigenvalues. The minimum is realized by the configuration in which a single edge is tangent to S (contributing the total tangential spin $j^{u+d} = \frac{1}{2}$) while another edge ends at the node on just one side ($j^u = \frac{1}{2}$, $j^d = 0$); this gives $8\pi\gamma\ell_{\mathrm{P}}^2\cdot\frac{1}{2}\sqrt{2\cdot\frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot\frac{3}{2}} = 2\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2$, exactly half of $\Delta A_\perp$. Thus the absolute minimal nonzero quantum of area in the full kinematical Hilbert space is $\Delta A_{\min} = 2\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2$; the transversal gap is twice this. We keep the distinction explicit to avoid the common conflation of the two.

Proof sketch. Partition S into cells S_I so fine that each carries at most one transversal edge. The regularized area is $\hat{A}(S) = \lim\sum_I\sqrt{\hat{E}^i(S_I)\hat{E}^i(S_I)}$. By Theorem 7.1, eq. (17), on a single transversal spin-j edge the summand $\hat{E}^i(S_I)\hat{E}^i(S_I)$ equals $(8\pi\gamma\ell_{\mathrm{P}}^2)^2\,j(j+1)$ times the identity on V_j, and it annihilates edges not meeting S_I. The square root is therefore already diagonal on V_j, with eigenvalue $8\pi\gamma\ell_{\mathrm{P}}^2\sqrt{j(j+1)}$; summing over the finitely many transversal edges gives eq. (18). Diagonality in the spin-network basis shows $\hat{A}(S)$ is symmetric with a complete orthonormal eigenbasis, hence essentially self-adjoint (its eigenvectors are a total set in the domain, so the deficiency indices vanish). Since the eigenvalues form a discrete set with finite multiplicities on each finite graph, the spectrum is pure point. For edges tangent to S or ending at nodes on S one uses the refined three-quantum-number formula below. □

Remark 7.4 (Full spectrum with nodes on the surface, [S]). When edges are tangent to S or a node lies on S, one splits the edges at a node into those crossing to the "up" side (j^u), the "down" side (j^d), and those tangent (j^{u+d} labelling the total tangential spin). The eigenvalue contribution of such a node is

$$8\pi\gamma\ell_{\mathrm{P}}^2\cdot\tfrac{1}{2}\sqrt{2j^u(j^u+1) + 2j^d(j^d+1) - j^{u+d}(j^{u+d}+1)}, \tag{20}$$

which reduces to eq. (18) for a single transversal edge ($j^u = j^d = j$, $j^{u+d} = 0$) [5]. The full area spectrum is the set of finite sums of such contributions. In this node-on-surface sector $\hat{A}(S)$ is diagonal only in an intertwiner basis *adapted to* S—one that recouples the up, down, and tangential edge subsets so that j^u, j^d, j^{u+d} are good quantum numbers; in a generic (unadapted) intertwiner basis the operator is not diagonal, though it remains self-adjoint with the same spectrum.

7.2 Reading of the result

Two features deserve emphasis. First, the spectrum is discrete and bounded below by a nonzero gap: there is a smallest quantum of area, of order $\gamma\ell_{\mathrm{P}}^2$. This is not imposed; it follows from the compactness of SU(2) (its irreducibles are finite-dimensional and labelled by a discrete set) through Peter–Weyl. The representation theory of a compact group is the origin of the discreteness of geometry. Second, the eigenvalue depends on γ: the Barbero–Immirzi parameter sets the physical area scale, and is a genuine quantization ambiguity, fixed externally (e.g. by matching black-hole entropy). We flag the identification "one transversal spin-$\frac{1}{2}$ edge = one Planck-scale area quantum of the physical world" as [H]: the operator and its spectrum are [S], but that they measure areas in our universe is a physical hypothesis.

8 Geometric operators II: volume

8.1 Construction and support

The classical volume of a region R is $V(R) = \int_R \sqrt{|\det E|}\, d^3x = \int_R \sqrt{\frac{1}{3!}|\epsilon_{ijk}\epsilon^{abc}E^a_i E^b_j E^c_k|}\, d^3x$. Regularizing and quantizing gives an operator $\hat{V}(R)$ that, unlike the area operator, acts at vertices and mixes intertwiners.

Proposition 8.1 (Volume operator: support on high-valence vertices, [S/H]). *The volume operator $\hat{V}(R)$ acts nontrivially only at vertices of the graph contained in R, and its action at a vertex v is a symmetric operator on the intertwiner space $\mathcal{K}_v$ built from the triple products $\hat{q}_{IJK} = \epsilon_{ijk}\hat{E}^i_I\hat{E}^j_J\hat{E}^k_K$ of flux operators on triples of edges (I, J, K) incident at v. It annihilates gauge-invariant vertices of valence ≤ 3, and its nonzero eigenvalues scale as ℓ_{P}^3 and are discrete.*

Proof sketch. The determinant of the triad is cubic in fluxes, so after regularization the operator is a sum of triple-grasping terms $\hat{q}_{IJK}$ acting on the three edges I, J, K at a common vertex; away from vertices the fluxes commute and the operator vanishes. On a trivalent gauge-invariant vertex, SU(2) invariance forces the three grasped generators to close (the sum of the three $\mathfrak{su}(2)$ generators annihilates the invariant), and antisymmetrization ϵ_{ijk} then makes $\hat{q}_{IJK}$ vanish; hence three-valent (and lower) vertices carry no volume. On higher-valence vertices $\hat{q}_{IJK}$ is a nonzero antisymmetric operator, and

$$\hat{V}(R) = \sum_{v\in R}\left|\frac{1}{8}\sum_{I<J<K}\kappa_{IJK}\hat{q}_{IJK}\right|^{1/2} \tag{21}$$

(up to regularization constants) is symmetric on the finite-dimensional $\mathcal{K}_v$, hence has a real discrete spectrum. See [6, 2]. □

8.2 Two inequivalent regularizations

There are two standard volume operators, and they are genuinely different operators, not two presentations of one:

- the *Rovelli–Smolin–De Pietri* (RS) operator sums the square roots of the absolute values of the individual triple grasps, so that at a vertex v

  $$\hat{V}_v^{RS} \;\propto\; \ell_{\mathrm{P}}^3 \sum_{I<J<K}\left|\hat{q}_{IJK}\right|^{1/2}, \tag{22}$$

  taking the absolute value (and root) grasp by grasp *before* summing;

- the *Ashtekar–Lewandowski* (AL) operator first forms the orientation-weighted sum and takes a single absolute value and root *after*,

$$\hat{V}_v^{AL} \propto \ell_{\mathrm{P}}^3 \Big| \tfrac{1}{8} \sum_{I<J<K} \kappa_{IJK}\, \hat{q}_{IJK} \Big|^{1/2}, \qquad \kappa_{IJK} = \operatorname{sgn}\det(\dot{e}_I, \dot{e}_J, \dot{e}_K), \tag{23}$$

  with κ_{IJK} the orientation sign of the edge tangents.

The AL operator is sensitive to the differential structure of the vertex (edge tangent orientations) through κ_{IJK}; the RS operator is not. They coincide on four-valent vertices but differ in general [6, 2].

Proposition 8.2 (Flux consistency selects AL, [S/H])**.** *Requiring that the quantized volume be consistent with an independent quantization of the triad via the flux operators (the Giesel–Thiemann consistency check) singles out the Ashtekar–Lewandowski operator; the Rovelli–Smolin operator fails this check.*

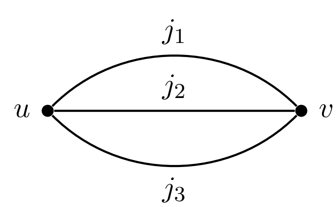
Discussion. Giesel and Thiemann compare the volume operator built directly from fluxes with the one obtained from an alternating-sum ("det") quantization and show the two agree in the AL case up to controlled ordering ambiguities but not in the RS case; see [15]. This is a consistency argument within the kinematical theory, not a physical measurement, so we tag the *selection* [S/H]: the mathematics ([S]) fixes the operator, while "AL volume is the physical volume of space" remains [H]. □

Remark 8.3. The existence of two inequivalent, physically motivated volume operators is a useful honesty check on the reconstruction program. The kinematical framework does not, by itself, force a unique geometry operator beyond area; additional input (a consistency criterion, or ultimately the dynamics) is needed. We do not paper over this.

9 Worked examples

9.1 The theta graph

Example 9.1 (Theta graph, [S])**.** Let Θ be the graph with two vertices u, v joined by three edges e_1, e_2, e_3, carrying spins j_1, j_2, j_3 (all oriented $u \to v$).



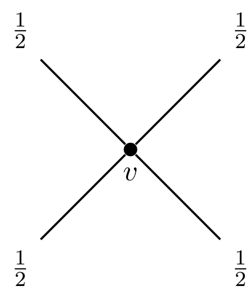
Each vertex is three-valent, so by Theorem 6.1 its intertwiner space is at most one-dimensional, and a gauge-invariant state on Θ exists iff

$$|j_1 - j_2| \le j_3 \le j_1 + j_2 \quad \text{and} \quad j_1 + j_2 + j_3 \in \mathbb{Z}. \tag{24}$$

When these hold, $\dim \mathcal{K}_u = \dim \mathcal{K}_v = 1$ and the spin network is unique up to phase; its state is the evaluation of the $3j$-symbols against the three Wigner matrices. Because both vertices are trivalent, $\hat{V} = 0$ on Θ: the theta graph has area (through any surface separating its edges) but no volume. It is the minimal illustration that in this framework area and volume are logically independent, representation-theoretic quantities.

9.2 The four-valent spin-$\frac{1}{2}$ vertex

Example 9.2 (Four spin-$\frac{1}{2}$ edges, [S]). Consider a single vertex with four incident edges all carrying $j = \frac{1}{2}$.



By eq. (15) with the pairing (12)(34), the intermediate spin k runs over $\{0, 1\}$ (both $V_{1/2} \otimes V_{1/2} = V_0 \oplus V_1$), so $\dim \mathcal{K}_v = 2$. Equivalently, by the character integral eq. (13) with $\chi_{1/2}(\theta) = 2\cos\theta$,

$$\dim \mathcal{K}_v = \frac{2}{\pi}\int_0^\pi (2\cos\theta)^4 \sin^2\theta \, d\theta = \frac{2}{\pi}\cdot 16 \int_0^\pi \cos^4\theta \sin^2\theta \, d\theta = \frac{2}{\pi}\cdot 16 \cdot \frac{\pi}{16} = 2. \tag{25}$$

This two-dimensional space carries the smallest nontrivial volume operator. In the recoupling basis $\{|k=0\rangle, |k=1\rangle\}$ the relevant grasping operator $\hat{q} = \hat{q}_{123}$ (the single independent triple-product for a four-valent vertex, the fourth edge being fixed by gauge invariance) is a Hermitian operator whose matrix elements are computed from $6j$-symbols. Since $\hat{q}$ is cubic in the fluxes, and each flux carries dimension ℓ_{P}^2, its natural scale is $c_0 = (8\pi\gamma\ell_{\mathrm{P}}^2)^3$, of dimension length^6. In the recoupling basis $\hat{q}$ is off-diagonal and takes the form

$$\hat{q} \;=\; c_0 \, \frac{\sqrt{3}}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{26}$$

an anti-diagonal imaginary matrix (it is i times a real antisymmetric matrix, as $\hat{q}$ is built from an odd number of grasps and an ϵ_{ijk}); it is Hermitian, with eigenvalues $\pm c_0 \frac{\sqrt{3}}{4}$. The volume operator $\hat{V} = \sqrt{|\hat{q}|}$ therefore has the single positive eigenvalue

$$V_{1/2^{\times 4}} \;=\; \Big(c_0 \, \tfrac{\sqrt{3}}{4}\Big)^{1/2} \;=\; \Big(\tfrac{\sqrt{3}}{4}\Big)^{1/2} (8\pi\gamma\ell_{\mathrm{P}}^2)^{3/2} \;=\; \Big(\tfrac{\sqrt{3}}{4}\Big)^{1/2} (8\pi\gamma)^{3/2} \, \ell_{\mathrm{P}}^3, \tag{27}$$

of order $\gamma^{3/2}\ell_{\mathrm{P}}^3$, with the correct dimension of length^3. We stress that this is an exact result once the regularization is fixed: for the Ashtekar–Lewandowski operator the eigenvalue is completely determined [6], and at valence four it agrees with the Rovelli–Smolin–De Pietri operator, whose four-valent matrix element $\frac{\sqrt{3}}{4}$ in eq. (26) was first computed by recoupling theory by De Pietri and Rovelli [14]. The residual "prefactor ambiguity" is therefore not a vagueness within a fixed operator but the genuine inequivalence between the RS and AL regularizations discussed in Section 8; each individually has an exact spectrum. This four-valent vertex is the elementary "quantum of three-volume": the smallest chunk of space that has volume at all. Note the structural point: volume, unlike area, mixes the intertwiner basis—it is genuinely an operator on $\mathcal{K}_v$, not a label—so a state of definite volume is a specific superposition of recoupling channels.

9.3 Area spectrum of a punctured surface

Example 9.3 (Surface pierced by n fundamental edges, [S])**.** Let S be pierced transversally by n edges each of spin $\frac{1}{2}$. By Theorem 7.2,

$$A = 8\pi\gamma\ell_{\mathrm{P}}^2 \sum_{i=1}^{n} \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)} = 8\pi\gamma\ell_{\mathrm{P}}^2 \cdot n \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2\, n. \tag{28}$$

The area grows in integer steps of the gap $\Delta A = 4\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2$; a macroscopic area of $1\,\mathrm{m}^2$ corresponds to $n \sim 10^{69}$ punctures at $\gamma \sim 1$. That such enormous puncture numbers are needed is exactly why the discreteness has escaped direct detection, and why the continuum limit (Section 14) is a serious question rather than a triviality.

10 Large-spin geometry and the classical shadow

The reconstruction thesis demands more than a Hilbert space: it must be possible, in some limit, to see a smooth Riemannian three-geometry emerge from the representation data. We record what is known and mark clearly where it stops being a theorem.

10.1 A single vertex as a quantum polyhedron

A four-valent gauge-invariant vertex with edge spins $j_1, \ldots, j_4$ has, by Minkowski's theorem, a classical counterpart: there is a unique convex polyhedron (here a tetrahedron) with four faces whose areas are $A_i \propto \sqrt{j_i(j_i+1)}$ and whose face normals sum to zero (the closure condition, which is exactly the Gauss constraint $\sum_i \vec{J}_i = 0$ in the classical limit). The intertwiner space $\mathcal{K}_v$ is the quantization of the space of such tetrahedra at fixed face areas, and the volume operator quantizes the tetrahedron's volume.

Proposition 10.1 (Closure is the Gauss constraint, [S])**.** *Let $\vec{J}_i$ be the angular-momentum vectors associated with the four edges at a four-valent vertex. Gauge invariance of the intertwiner is equivalent to the operator closure condition $\big(\sum_{i=1}^4 \vec{J}_i\big)\iota_v = 0$. In the large-spin regime the expectation values $\langle \vec{J}_i \rangle$ satisfy $\sum_i \langle \vec{J}_i \rangle = 0$, which is Minkowski's closure condition for a tetrahedron with face-area vectors $\langle \vec{J}_i \rangle$.*

Proof. The generator of gauge transformations at v is $\sum_i \vec{J}_i$ (the diagonal $\mathfrak{su}(2)$ action on the tensor product), and ι_v is gauge invariant iff it is annihilated by this generator, i.e.

$\sum_i \vec{J}_i \, \iota_v = 0$. Taking expectation values in a state peaked on classical face vectors gives $\sum_i \langle \vec{J}_i \rangle = 0$, Minkowski's condition, which for four vectors with given lengths determines a tetrahedron up to rotation. See [17]. □

This is the sharpest sense in which "the intertwiner is a quantum polyhedron": the gauge constraint is geometric closure, and the recoupling parameter k of Theorem 9.2 is the quantized dihedral-angle/diagonal variable of the tetrahedron.

10.2 Coherent states and the honest gap

To assemble many vertices into an approximate smooth geometry one uses Livine–Speziale or heat-kernel coherent states, spin-network superpositions peaked on classical holonomy–flux data. In controlled cases (single vertices, small graphs, symmetric reductions such as loop quantum cosmology) these reproduce the expected classical areas, volumes, and dihedral angles to leading order in $1/j$. What is *not* available is a theorem that a coarse-grained large-spin limit of the full theory reproduces a generic solution of Einstein's equations, with control over the errors and the refinement. The coarse-graining/renormalization program [18, 19] is precisely the attempt to close this gap, and recent tensor-network methods [21] make it numerically tractable in restricted settings, but the general statement remains open. We therefore tag the claim "large-spin spin networks reconstruct a smooth three-geometry" as [H], and the stronger "…reconstruct a solution of general relativity" as [H] and open. This is the single most important honesty point of the paper, and Section 14 returns to it.

11 Uniqueness of the representation

A natural worry is that the whole construction depends on the ad hoc choice of the AL measure. It does not: the representation is forced.

Theorem 11.1 (LOST uniqueness, [S])**.** *Among all representations of the holonomy–flux $*$-algebra $\mathfrak{a}$ by operators on a Hilbert space that (i) carry a unitary action of the spatial diffeomorphism group under which the vacuum is invariant, (ii) are cyclic for that vacuum, and (iii) satisfy natural regularity/continuity conditions, there is exactly one, up to unitary equivalence: the Ashtekar–Isham–Lewandowski representation on $\mathcal{H}_{\mathrm{kin}} = L^2(\overline{\mathcal{A}}, \mu_{\mathrm{AL}})$.*

This is the theorem of Lewandowski, Okołów, Sahlmann, and Thiemann [7], with an independent proof for the Weyl (C*-algebraic) form of the algebra by Fleischhack [8]. Its force for the reconstruction thesis is this: once one demands background independence (diffeomorphism covariance) at the quantum level, the kinematical arena is not a modelling choice but a theorem. The representation labels are not merely *a* way to encode quantum geometry; they are the unique diffeomorphism-invariant way. We tag Theorem 11.1 [S]: it is a proved statement about a well-defined algebra. Its physical reading, "therefore nature's quantum geometry is a spin network," composes it with an [H] target-identification and is [H].

Remark 11.2 (Scope of the theorem)*.* The theorem assumes a compact structure group and analytic (or semianalytic) graphs, and it constrains the *kinematical* representation only. It says nothing about the dynamics, and it does not by itself resolve the continuum limit. The uniqueness is of the arena, not of the physics played in it.

12 Relation to the other regimes and the S/H/P calculus

The spin-network regime is one of twelve in a modular library. It is not asserted to subsume or be subsumed by the others; the correct way to combine two regimes is to compose the specific claims used, and the composite inherits the weakest warrant (Theorem 2.1). We record the main adjacencies.

12.1 Classical geometry (QG-I)

Spin networks are meant to reconstruct a semiclassical Cauchy slice of a Lorentzian spacetime [23]. The bridging claim—that a large-spin/coherent-state family of spin networks approximates a given smooth Riemannian three-geometry—uses the kinematical theorems ([S]) together with a semiclassical-limit hypothesis ([H]). By the calculus the composite is [H], *not* [S/H], even though both endpoints are individually [S/H]: the bridge is weaker than either end. This is the disciplined content of "spin networks reconstruct classical geometry" and it is why we do not claim more.

12.2 Gauge and constraints (QG-II)

The Gauss constraint's SU(2) invariance at each vertex (Theorem 6.1) is the graph-local avatar of the general gauge principle: the physical vertex data is the intertwiner space $\mathcal{K}_v$, the "retained isotropy" at v, not a coarse quotient. In the homological (BV–BRST) language of the gauge regime, $\mathcal{K}_v$ is the degree-zero cohomology of the local Gauss complex. The two descriptions agree where they overlap; both are [S/H].

12.3 Spin-foam dynamics (QG-III)

The covariant dynamics is supplied by spin foams: a spin foam is a two-complex interpolating between an incoming and an outgoing spin network, and it assigns a transition amplitude

$$Z(\mathcal{F}):\ \mathcal{H}_{\Gamma_{\rm in}} \to \mathcal{H}_{\Gamma_{\rm out}}, \qquad \mathcal{H}_\partial = \bigoplus_{\Gamma} \mathcal{H}_\Gamma. \tag{29}$$

The boundary Hilbert space $\mathcal{H}_\partial$ built here is exactly the domain and codomain of $Z(\mathcal{F})$ [16, 17]. The spin-foam dynamics carries bare status [H] (its continuum/refinement limit is open), so any claim "spin networks plus spin foams give a complete quantum-gravity dynamics" composes [S/H] kinematics with [H] dynamics and is [H].

$$\begin{tikzcd}
\mathcal{H}_{\Gamma_{\rm in}} \arrow[r, "Z(\mathcal{F})"] \arrow[d, hook, "\text{basis}"'] & \mathcal{H}_{\Gamma_{\rm out}} \arrow[d, hook, "\text{basis}"] \\
\{(\Gamma_{\rm in}, j_e, \iota_v)\} \arrow[r, dashed, "\text{sum over 2-complexes}"'] & \{(\Gamma_{\rm out}, j_e, \iota_v)\}
\end{tikzcd}$$

12.4 Explicit non-identifications

We do *not* identify spin-network quantum geometry with the spectral-triple (noncommutative geometry) reconstruction of spacetime, nor with the causal-set substrate; these are separate candidate reconstructions of the same classical regime, not of each other. Merging them would require an explicit new argument and a fresh status derivation, and none is claimed here. This is the modularity discipline of the library: partially overlapping regimes, not a monolithic unification.

13 A Haskell model of the representation data

To make the reconstruction concrete and checkable, we accompany the paper with a small Haskell development modelling the core structures: graphs with SU(2) representation labels, intertwiner dimension counting via recoupling, and area eigenvalues. The design mirrors the executable status calculus of the library's formalization layer.

The intertwiner count for a four-valent vertex is computed both by the recoupling formula eq. (15) and, as a cross-check, by a direct Clebsch–Gordan fusion of the four representations; the module carries a property asserting the two agree for all spins up to a cutoff. The area eigenvalue eq. (18) is a pure function of the piercing spins, and a checkable property asserts monotonicity in each spin and the value $4\sqrt{3}\,\pi\gamma\ell_{\mathrm{P}}^2$ for a single spin-$\frac{1}{2}$ puncture. These are modest checks, deliberately: they verify the representation-theoretic bookkeeping, which is the [S] content, and make no claim about the [H] physics. The core module is reproduced in Section A; the complete sources (four modules: `Core`, `SpinNetwork`, `Properties`, `Main`) accompany this submission as ancillary files.

14 Limitations and open problems

We state the honest status of the program.

Continuum/semiclassical limit (open). The central open problem for this regime is whether, and in what precise sense, a coarse-grained or large-spin limit of spin-network states reproduces a smooth Riemannian three-geometry, and ultimately whether the full (kinematics plus dynamics) theory has a continuum limit reproducing general relativity. Coherent-state and coarse-graining techniques exist [18, 19, 20], and recent tensor-network numerical methods have made the refinement question computationally tractable in restricted settings [21], but a theorem establishing that QG-I is the correct classical shadow of this regime does not exist. We tag the semiclassical-limit claim [H] and flag it as the primary unresolved item.

The Barbero–Immirzi ambiguity. The area and volume spectra scale with γ, a dimensionless parameter not fixed by the kinematics. It is usually fixed by matching black-hole entropy, which is itself a semiclassical ([H]) input. That a purely kinematical ambiguity is fixed only by a semiclassical argument is a genuine feature to be aware of, not a bug we conceal.

Volume operator ambiguity. As in Section 8, two inequivalent volume operators exist; the flux consistency check selects AL, but this is a mathematical criterion, and the ultimate selection should come from the dynamics.

Dynamics. The Hamiltonian constraint operator that would generate time evolution is not unique and its physical spectrum is not under full control; recent numerical work on its graph-changing action [22] is progress, not closure. The present paper is deliberately confined to kinematics, where the results are theorems.

15 Discussion

The reconstruction thesis is easy to state and hard to earn. In the spin-network regime it is earned, for the kinematics, by genuine theorems: the kinematical Hilbert space exists and is unique (LOST), the spin-network states are an orthonormal basis (Peter–Weyl), and geometry appears as the discrete spectrum of self-adjoint (or symmetric) operators built from SU(2) representation labels (area, volume). At this level "the representation labels are the geometry" is not a metaphor: there is no metric in the definition of a spin-network state, only a graph, spins, and intertwiners, and area/volume are computed from them.

What the thesis has not yet earned, in this regime, is the physics: the passage from this combinatorial-representation-theoretic arena to the smooth spacetime of general relativity, and the dynamics that would make it a theory of gravity rather than a theory of quantum geometry alone. We have marked every such passage [H] and left the continuum limit explicitly open. The value of the status calculus is exactly that it prevents the settled kinematics from lending unearned credibility to the unsettled physics: a chain of reasoning is only as strong as its weakest warrant.

Seen this way, spin networks are the sharpest available example, within the twelve regimes, of reconstructing *spatial geometry* from invariant representation data—and simultaneously an honest illustration of how far such a reconstruction can be pushed rigorously before physical hypotheses must be added. The neighbouring regimes fill in the rest of the picture: spin foams supply the dynamics on these boundary states, the gauge/BV–BRST regime supplies the homological reading of the Gauss constraint, and the classical regime supplies the target that the still-missing continuum limit must reproduce.

16 Conclusion

We have presented quantum spatial geometry as representation-theoretic data on a graph and made the reconstruction thesis concrete and checkable in its kinematical part. The main proved statements—the AL measure and kinematical Hilbert space (Theorem 4.2), the spin-network basis (Theorem 5.2), the Gauss quotient as the intertwiner space (Theorem 6.1), the area operator's self-adjointness and discrete spectrum with an area gap (Theorem 7.2), the volume operator's support on high-valence vertices and the AL/RS inequivalence (Theorems 8.1 and 8.2), and the LOST uniqueness of the representation (Theorem 11.1)—are all [S]: theorems in a well-defined mathematical setting. The physical readings—that these structures are the quantum geometry of our universe and that a smooth three-metric re-emerges in a large-spin limit—are [H] and, in the case of the continuum limit, openly unresolved.

The regime composes modularly with its neighbours through the weakest-link status calculus, and the accompanying Haskell model turns the representation-theoretic bookkeeping into machine-checkable properties. The honest headline is narrow and firm: at the kinematical level, in this regime, geometry is representation data, and this is a theorem, not a hope. The broad headline—that this reconstructs the geometry of the physical world—remains a hypothesis, and we have been careful to say so.

A The core Haskell module

For reproducibility we reproduce the central module, which models SU(2) irreducibles, Clebsch–Gordan fusion, intertwiner-space dimensions (by two independent routes), and the area eigenvalue. It compiles with GHC and underlies the checkable properties of Section 13. The remaining modules (`SpinNetwork`, `Properties`, `Main`) accompany the submission as ancillary files.

```
-- SU(2) representation-theoretic core for spin networks (module Core).
-- All content here is the [S] (standard mathematical) bookkeeping.

newtype Spin = Spin (Ratio Int) deriving (Eq, Ord)

-- Smart constructor: reject negatives and non-half-integers.
spin :: Ratio Int -> Spin
spin r
  | r < 0                    = error ("spin: negative " ++ show r)
  | denominator (2 * r) /= 1 = error ("spin: not a half-integer " ++ show r)
  | otherwise                = Spin r

-- Dimension of V_j = 2j+1.
dimIrrep :: Spin -> Int
dimIrrep (Spin r) = numerator (2 * r) + 1

-- SU(2) Casimir eigenvalue j(j+1) on V_j.
casimir :: Spin -> Ratio Int
casimir (Spin r) = r * (r + 1)

-- Triangle / admissibility for a trivalent coupling j1 (x) j2 -> j3.
triangle :: Spin -> Spin -> Spin -> Bool
triangle (Spin a) (Spin b) (Spin c) =
  abs (a - b) <= c && c <= a + b && denominator (a + b + c) == 1

-- Clebsch-Gordan fusion V_a (x) V_b = (+)_{k=|a-b|}^{a+b} V_k.
fuse :: Spin -> Spin -> M.Map Spin Int
fuse (Spin a) (Spin b) =
  M.fromListWith (+)
```

```
    [ (Spin k, 1)
    | let lo = abs (a - b), let hi = a + b
    , k <- takeWhile (<= hi) (iterate (+ 1) lo) ]

-- Iterated fusion of a list of irreducibles, with multiplicities.
fuseMany :: [Spin] -> M.Map Spin Int
fuseMany []       = M.singleton (spin 0) 1
fuseMany (x : xs) = foldl step (M.singleton x 1) xs
  where step acc j = M.fromListWith (+)
          [ (k, m * mult)
          | (Spin a, m) <- M.toList acc
          , (k, mult)   <- M.toList (fuse (Spin a) j) ]

-- dim Inv(V_{j1} (x) ... (x) V_{jn}) = multiplicity of spin 0.
intertwinerDim :: [Spin] -> Int
intertwinerDim js = M.findWithDefault 0 (spin 0) (fuseMany js)

-- Independent four-valent recoupling count (cross-check).
intertwinerDim4 :: Spin -> Spin -> Spin -> Spin -> Int
intertwinerDim4 (Spin a) (Spin b) (Spin c) (Spin d) =
  length
    [ () | let lo = max (abs (a - b)) (abs (c - d))
         , let hi = min (a + b) (c + d), lo <= hi
         , k <- takeWhile (<= hi) (iterate (+ 1) lo)
         , denominator (a + b + k) == 1, denominator (c + d + k) == 1 ]

-- Area eigenvalue in units of 8 pi gamma ell_P^2: sum_i sqrt(j_i(j_i+1)).
areaEigenvalue :: [Spin] -> Double
areaEigenvalue js = sum [ sqrt (toDouble (casimir j)) | j <- js ]
  where toDouble r = fromIntegral (numerator r) / fromIntegral (denominator r)
```

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