Covariant Quantum Geometry as a State Sum: Spin Foams, Vertex Amplitudes, and the Refinement Problem of Boundary Amplitudes

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Abstract

A spin foam is a two-complex whose faces and edges carry representation labels and whose state sum defines a transition amplitude between spin-network boundary states. We treat this construction as the covariant, sum-over-histories layer of a reconstruction program in which spacetime, causality, and measurement are read off invariant representation structures rather than quantized on a fixed background. The object of interest is not a metric but a triple: a boundary Hilbert space, an amplitude map defined by a colored two-complex, and the behavior of that map under refinement of the complex. We give a self-contained account of colored two-complexes dual to triangulated four-manifolds, the face/edge/vertex decomposition of the state sum $\mathcal{Z}(\mathcal{F}) = \sum_{\{j_f\},\{\iota_e\}} \prod_f A_f \prod_e A_e \prod_v A_v$, and the constraint route from topological BF theory to gravity: strong simplicity in the Barrett–Crane model and weak (linear) simplicity through the Engle–Pereira–Rovelli–Livine Y_γ map in the model that carries the Barbero–Immirzi parameter. Our results are stated as labeled propositions with each claim tagged by the project's S/H/P epistemic calculus. That the truncated state sum is a well-defined finite amplitude map, that the topological BF sum is a triangulation invariant after quantum-group regularization, and that the Engle–Pereira–Rovelli–Livine four-simplex vertex has a large-spin asymptotic proportional to the cosine of the Regge action are standard mathematics ([S]). That the resulting vertex is the correct gravitational dynamics, and above all that the family of amplitudes over refinements has a continuum limit reproducing general relativity, are heuristic ([H]) and, in the case of the continuum limit, openly unresolved. We are explicit about the flatness problem, the accidental curvature constraint that obstructs a naive large-spin recovery of curved geometry, and about the effective-spin-foam reframing that partially addresses it. A worked example computes the vertex structure dual to a single four-simplex and states the asymptotic formula; a companion Haskell development exhibits the refinement invariant exactly in the solvable topological case and supplies checkable recoupling bookkeeping. We close by placing spin foams among the other regimes of the library:

spin networks supply the boundary data, classical Lorentzian geometry is the target the missing continuum limit must reproduce, and the weakest-link status calculus bounds every such composite by its bare-[H] spin-foam link.

Contents

1 Introduction

1.1 The reconstruction thesis for covariant dynamics

The canonical picture of loop quantum gravity gives a kinematics: a state of quantum three-geometry is a spin network, a graph whose edges carry SU(2) representations and whose vertices carry intertwiners, and geometry appears as the discrete spectrum of area and volume operators built from those labels. That picture is the subject of a companion paper. What it does not give, on its own, is dynamics. A theory of gravity must say how one three-geometry evolves into another, and general covariance forbids doing so by a preferred time coordinate. The covariant answer is a sum over histories: instead of evolving a spin network by a Hamiltonian, one sums over all "quantum four-geometries" interpolating between a fixed incoming and outgoing spin network. A spin foam is precisely such a history, and the sum over them is the spin foam amplitude.

This paper works in that covariant layer, under the organizing principle shared across a modular library of twelve quantum-gravity regimes:

> *Quantum gravity is not primarily the quantization of material objects in spacetime. It is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures.*

In the present regime the invariant structure is a colored two-complex. A history is a two-complex $\mathcal{F}$ whose faces are labeled by irreducible representations and whose edges are labeled by intertwiners; the amplitude of that history is a product of local recoupling invariants, one per face, edge, and vertex; and the physical content is the boundary amplitude obtained by summing over the bulk labels. Nowhere in this data is there a spacetime metric. A smooth four-geometry, with its Einstein–Hilbert or Regge action, is not put in by hand: it is recovered, if at all, only in an asymptotic (large-spin) limit of the representation-theoretic sum. Whether that recovery survives refinement of the two-complex into a genuine continuum theory is the central open problem of the field, and we treat it as open throughout.

1.2 What is settled and what is not

Honesty about status is the point of this series, so we state it at the outset. The following are theorems or standard constructions in a well-defined mathematical setting, and we tag them [S]:

- For a finite two-complex with a cutoff on the representation labels, the state sum is a finite sum of products of finite-dimensional recoupling invariants, hence a well-defined amplitude map between boundary Hilbert spaces (Theorem 3.4).
- The topological BF state sum built from the $\{15j\}$ symbol is invariant under Pachner moves once regularized by a quantum group, so it is a piecewise-linear invariant of the triangulated four-manifold (Theorem 4.1).
- Amplitudes glue along shared boundary spin networks, giving the state sum the composition law of a functor on a cobordism-like category (Theorem 3.5).
- The Engle–Pereira–Rovelli–Livine four-simplex vertex amplitude, evaluated on boundary data peaked on a nondegenerate Regge geometry, has a large-spin stationary-phase asymptotic whose leading term is proportional to $\cos(S_{\text{Regge}})$ (Theorem 5.1).

The following are physical hypotheses, well motivated but not proved, and we tag them [H]:

- That the Engle–Pereira–Rovelli–Livine or Freidel–Krasnov vertex is the correct quantum dynamics of gravity. The vertex is fixed by imposing a discretized simplicity constraint, but the choice of how to impose it (strongly, weakly, on which variables), the face amplitude, and the measure are physical inputs, not consequences of an axiom.
- That summing over refinements of the two-complex, or over two-complexes themselves, defines a continuum theory that reproduces general relativity. This is open. The known obstruction, the flatness problem (Section 7.2), is that a naive large-spin limit at fixed complex is dominated by flat geometries.

The dictionary of the library assigns the entry *spin foam two-complex with face/edge labels* the bare status [H], the only bare-[H] entry among the core geometric regimes, precisely because the dynamics and its continuum limit remain a heuristic proposal rather than a settled structure. We honor that label: the abstract and this introduction state the continuum-limit status as unresolved, and every claim in the body carries its warrant explicitly.

1.3 Contributions and outline

We do not claim new physics. The contribution is a rigorous, status-labeled synthesis of the covariant-dynamics regime organized around one invariant — the boundary amplitude and its refinement behavior — together with a companion computation that exhibits that invariant exactly in the solvable topological case.

Section 2 fixes the reconstruction pipeline and the status calculus. Section 3 defines colored two-complexes dual to triangulations, the face/edge/vertex data, and the state sum, and states the well-definedness and gluing results. Section 4 gives the constraint route from BF theory to gravity: the topological BF state sum and its invariance, the simplicity constraints, the Barrett–Crane model, and the Y_γ map defining the Engle–Pereira–Rovelli–Livine vertex. Section 5 states the semiclassical asymptotics. Section 6 works the four-simplex example explicitly. Section 7 treats refinement, cylindrical consistency, the flatness problem, and the effective-spin-foam reframing honestly. Section 8 locates the regime among the others through the weakest-link calculus. Section 9 describes the Haskell model, whose core module is reproduced in Section A. Section 10 lists the open problems, and Sections 11 and 12 conclude.

2 Mathematical framework: reconstruction from representation data

2.1 The realization pipeline

The library models every physical representation as a translation from a mathematical object M to a physical target, factored through a realization:

$$M \longmapsto \Phi(M) \longmapsto \mathrm{Real}_\alpha(\Phi(M)) \longmapsto \text{physical representation} \longmapsto \text{Obs.}$$

For spin foams the mathematical object M is a colored two-complex; the intermediate $\Phi(M)$ is its state sum, a number attached to boundary data; the realization Real_α chooses a model (which group, which vertex amplitude, which face amplitude and measure); and the observable is the boundary transition amplitude $\mathcal{Z}(\mathcal{F}) : \mathcal{H}_{\Gamma_{\mathrm{in}}} \to \mathcal{H}_{\Gamma_{\mathrm{out}}}$. The index α matters: Barrett–Crane, Engle–Pereira–Rovelli–Livine, and Freidel–Krasnov are different realizations of the same abstract state-sum object, and they differ in physically consequential ways. Keeping α explicit is what prevents the honest statement "the topological state sum is a triangulation invariant" from silently upgrading into the false statement "the gravitational state sum is a continuum theory."

2.2 The status calculus

We use three ordered warrants $\mathsf{S} < \mathsf{H} < \mathsf{P}$: S for a standard mathematical or mathematical-physics correspondence, H for a strong heuristic physical representation, and P for a speculative ontological extension. A status is a tuple rendered [S], [H], [P], or a composite such as [S/H]; composition takes the worst component.

Definition 2.1 (Weakest-link composition)**.** For warrants a, b with $\mathsf{S} < \mathsf{H} < \mathsf{P}$, define $a * b = \max(a, b)$. A reasoning chain that composes representations of statuses $a_1, \ldots, a_n$ carries status $a_1 * \cdots * a_n$. Composition is monotone and S is a unit: $\mathsf{S} * a = a$.

The rule is not decoration. It is the precise sense in which the library is modular rather than unified: chaining spin-network kinematics (status [S/H]) to spin-foam dynamics (status [H]) yields a composite bounded above by [H], no matter how many rigorous theorems live inside the kinematical factor. One cannot buy credibility for the dynamics with the rigor of the kinematics.

2.3 The relevant dictionary entries

Three entries of the twenty-six-row dictionary bear on this paper.

Spin network: graph labeled by SU(2) *representations* = [S/H]. The boundary data. The kinematical Hilbert space and the area/volume spectra are [S] theorems; the physical identification with the geometry of our universe is [H].

Spin foam two-complex with face/edge labels = [H]. The subject of this paper. No assumptions field is populated; the limitation clause reads "continuum/ refinement limit and phenomenology are active research problems."

Cobordism category = [S/H]. The categorical grammar in which a spin foam is a morphism from an incoming to an outgoing spin network, and gluing is composition.

The composite of the first two, under Theorem 2.1, is [H]: a complete covariant quantum-gravity dynamics built from spin-network kinematics plus spin-foam amplitudes inherits the bare-[H] label of the dynamics. This is exactly why the roadmap's task for this module demands "record entries with explicit continuum-limit limitations" rather than asserting the package as settled.

2.4 Gauge and boundary as retained structure

A recurring structural fact across the library is that physical content is a groupoid, not a coarse quotient: gluing of local data is by equivalence, so automorphism and stabilizer data (gauge redundancy, symmetry) is retained rather than erased. In the spin-foam regime this appears twice. First, the edge intertwiners are exactly the gauge-invariant data at a node, the SU(2) (or $\mathrm{SL}(2,\mathbb{C})$) invariant subspace of a tensor product of representations; the state sum integrates over the group at each edge, which is the discrete avatar of imposing the Gauss constraint. Second, the boundary of a spin foam is a spin network taken modulo spatial diffeomorphisms, and the amplitude is a pairing on that quotient. The reconstruction never works with a metric; it works with representation labels and invariants, and the geometry is a derived quantity.

3 Colored two-complexes and the state sum

3.1 Two-complexes dual to triangulations

Definition 3.1 (Two-complex)**.** A (combinatorial, oriented) two-complex $\mathcal{F}$ is a set of *vertices* V, *edges* E, and *faces* F, together with incidence data: each edge has a source and target vertex, and each face is bounded by a cyclic sequence of edges. We write $f \ni e$ when the edge e lies on the boundary of the face f, and $e \ni v$ when the vertex v is an endpoint of e. The *boundary* $\partial\mathcal{F}$ is the graph formed by the boundary edges and vertices (those lying on a single face on one side).

The spin foams relevant to four-dimensional gravity arise as the two-skeleton of the cell complex dual to a triangulation Δ of a four-manifold $\mathcal{M}$. Under the duality,

$$\text{4-simplices of } \Delta \;\leftrightarrow\; \text{vertices } v, \quad \text{tetrahedra} \;\leftrightarrow\; \text{edges } e, \quad \text{triangles} \;\leftrightarrow\; \text{faces } f.$$

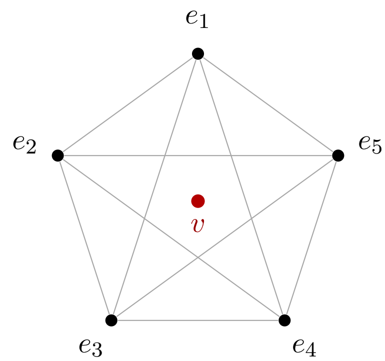

Figure 1: The local structure at a spin-foam vertex v dual to a four-simplex: five edges $e_1, \dots, e_5$ (dual to the five tetrahedra) meeting at v, with the ten faces (dual to the ten triangles) running as the ten links of the complete graph K_5. The vertex amplitude contracts the five edge intertwiners along this graph, giving a $\{15j\}$-type invariant.

A four-simplex has five tetrahedra and ten triangles on its boundary; dually, each vertex v is met by five edges and ten faces, and the local combinatorics at v is the complete graph K_5 whose five nodes are the foam edges and whose ten links are the foam faces. This K_5 pattern is what fixes the vertex amplitude to be a $\{15j\}$-type recoupling invariant. When $\mathcal{M}$ has boundary, $\partial\mathcal{F}$ is a pair of graphs $\Gamma_{\text{in}}, \Gamma_{\text{out}}$, the spin networks between which the foam interpolates.

3.2 Face, edge, and vertex labels

Definition 3.2 (Coloring)**.** A *coloring* of $\mathcal{F}$ assigns to each face f an irreducible unitary representation and to each edge e an intertwiner. In the Riemannian models the face label is an irreducible ρ_f of SU(2) (equivalently a spin $j_f \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, carrier V_{j_f} of dimension $2j_f + 1$) or of $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$; in the Lorentzian models it is a unitary principal-series representation of $\mathrm{SL}(2, \mathbb{C})$, labeled by a pair (ρ_f, k_f) with $\rho_f \in \mathbb{R}$ and $k_f \in \frac{1}{2}\mathbb{Z}$. The edge label ι_e is an element of the invariant space

$$\mathrm{Inv}\Big(\bigotimes_{f \ni e} V_{j_f}\Big) \;=\; \mathrm{Hom}_G\Big(\mathbb{C}, \bigotimes_{f \ni e} V_{j_f}\Big), \qquad G \in \{\mathrm{SU}(2), \mathrm{SL}(2, \mathbb{C})\},$$

that is, an intertwiner among the representations on the faces meeting the edge.

The invariant space is finite-dimensional for SU(2); for $\mathrm{SL}(2, \mathbb{C})$ it is infinite-dimensional, and the models handle this by an embedding (the Y_γ map of Section 4.3) that lands the SU(2) intertwiner inside an $\mathrm{SL}(2, \mathbb{C})$ one. On the boundary, the face and edge labels restrict to the edge and vertex labels of the boundary spin networks $\Gamma_{\text{in}}, \Gamma_{\text{out}}$: a boundary face is dual to a boundary spin-network edge, and a boundary edge is dual to a boundary node.

3.3 The state sum

Definition 3.3 (State-sum amplitude)**.** Fix a model, i.e. a choice of face amplitude A_f, edge amplitude A_e, and vertex amplitude A_v. For a two-complex $\mathcal{F}$ with fixed boundary coloring

$b = (j_\partial, \iota_\partial)$, the *state-sum amplitude* is

$$\mathcal{Z}(\mathcal{F})[b] \;=\; \sum_{\{j_f\},\{\iota_e\}} \prod_{f\in F} A_f(j_f) \prod_{e\in E} A_e(j_{f\ni e}, \iota_e) \prod_{v\in V} A_v\big(j_{f\ni v}, \iota_{e\ni v}\big), \tag{1}$$

where the sum runs over all colorings of the bulk (interior) faces and edges compatible with the fixed boundary b, and the products run over faces, edges, and vertices. As b varies, (1) is a kernel that defines a linear map

$$\mathcal{Z}(\mathcal{F}) \;:\; \mathcal{H}_{\Gamma_{\rm in}} \longrightarrow \mathcal{H}_{\Gamma_{\rm out}},$$

between the spin-network boundary Hilbert spaces.

The face amplitude is most often $A_f(j_f) = \dim V_{j_f} = 2j_f + 1$ (the BF choice), though the correct power of the dimension is itself a model choice that affects the divergence structure and the semiclassical limit. The vertex amplitude A_v is the crucial dynamical input; the entire content of a spin-foam model is the choice of A_v, which is fixed by imposing a discretized simplicity constraint on a topological BF theory, as we now describe.

Theorem 3.4 (Well-definedness of the truncated amplitude, [S])**.** *Let $\mathcal{F}$ be a finite two-complex and fix a cutoff Λ on the representation labels (for* SU(2)*: $j_f \le \Lambda$; for* SL(2, ℂ)*: a compact set of principal-series labels). Then for any fixed boundary coloring b, the truncated state sum $\mathcal{Z}_\Lambda(\mathcal{F})[b]$ obtained by restricting the sum in* (1) *to labels below Λ is a finite sum of finite products of finite-dimensional recoupling invariants, hence an absolutely convergent, well-defined complex number. The map $\mathcal{Z}_\Lambda(\mathcal{F}) : \mathcal{H}_{\Gamma_{\rm in}} \to \mathcal{H}_{\Gamma_{\rm out}}$ is linear and bounded on the finite-dimensional boundary label sectors.*

Proof. With $\mathcal{F}$ finite there are finitely many faces, edges, and vertices. With the cutoff, each bulk face ranges over finitely many labels and each bulk edge over a finite-dimensional intertwiner space (finite-dimensional after the Y_γ embedding in the $\mathrm{SL}(2,\mathbb{C})$ case), so the index set of the sum is finite. Each factor A_f, A_e, A_v is a contraction of finite-dimensional intertwiners — a Clebsch–Gordan or $\{nj\}$ recoupling coefficient — and hence a finite complex number. A finite sum of finite products of finite numbers is finite. Linearity in the boundary data is immediate from (1), which is multilinear in the boundary intertwiners; boundedness on a fixed finite-dimensional boundary sector is automatic. □

Theorem 3.4 is deliberately modest. It says the amplitude is a well-defined number *once a complex and a cutoff are fixed.* It says nothing about removing the cutoff (the sum over bulk spins may diverge; see Section 7) or about refining the complex. Those are exactly the open questions, and separating the settled part from them is the discipline the status calculus enforces.

3.4 Gluing and the composition law

Proposition 3.5 (Gluing, [S/H])**.** *Suppose $\mathcal{F} = \mathcal{F}_1 \cup_\Gamma \mathcal{F}_2$ is obtained by gluing two two-complexes along a common boundary spin-network graph Γ, so that $\partial\mathcal{F}_1 = \Gamma_{\rm in} \sqcup \Gamma$ and*

$\partial\mathcal{F}_2 = \Gamma \sqcup \Gamma_{\text{out}}$. *Then, with the natural face and edge amplitudes on the shared boundary distributed to avoid double counting,*

$$\mathcal{Z}(\mathcal{F}) = \sum_{c(\Gamma)} \mathcal{Z}(\mathcal{F}_1)\big[\,\cdot\,, c(\Gamma)\big]\ \mathcal{Z}(\mathcal{F}_2)\big[c(\Gamma),\,\cdot\,\big], \tag{2}$$

where the sum runs over colorings $c(\Gamma)$ *of the shared boundary. Equivalently,* $\mathcal{Z}(\mathcal{F}) = \mathcal{Z}(\mathcal{F}_2)\circ\mathcal{Z}(\mathcal{F}_1)$ *as maps of boundary Hilbert spaces. The assignment* $\Gamma \mapsto \mathcal{H}_\Gamma$*,* $\mathcal{F} \mapsto \mathcal{Z}(\mathcal{F})$ *is a functor from a cobordism-like category of spin networks and two-complexes to (finite-dimensional) vector spaces.*

Proof sketch. The state sum (1) factorizes over the two pieces: every bulk face, edge, and vertex belongs to exactly one of $\mathcal{F}_1, \mathcal{F}_2$, while the shared faces and edges lie on Γ. Fixing the shared coloring $c(\Gamma)$ makes each piece an independent state sum with Γ as (part of its) boundary, and summing over $c(\Gamma)$ reconstitutes the sum over the interior labels of $\mathcal{F}$ that lie on the gluing surface. Reindexing gives (2). The tag is [S/H] rather than [S]: for the topological BF models this is a genuine gluing axiom of an extended topological field theory ([S]), but for the gravitational models the boundary Hilbert spaces, the correct distribution of face amplitudes across the seam, and the convergence of the sum over $c(\Gamma)$ are model-dependent, which is the heuristic part. □

Theorem 3.5 is the structural reason spin foams sit in the cobordism/TQFT column of the dictionary. A spin foam is a morphism $\Gamma_{\text{in}} \to \Gamma_{\text{out}}$; composition is gluing; the identity is a trivial cylinder. The qualifier "like" matters: a genuine TQFT would assign a finite-dimensional space to each boundary and be diffeomorphism (indeed triangulation) invariant, and the gravitational models are not triangulation invariant — if they were, they would be topological and contain no local degrees of freedom, hence no gravitons. The tension between wanting a triangulation invariant (for a background-independent continuum limit) and wanting local curvature degrees of freedom is the conceptual core of the refinement problem (Section 7).

4 From BF theory to gravity: simplicity constraints

4.1 BF theory and its state sum

The starting point is a topological field theory. Let G be a Lie group with Lie algebra $\mathfrak{g}$. BF theory has fields a connection A (with curvature $F(A)$) and a $\mathfrak{g}$-valued two-form B, and action

$$S_{BF}[A,B] = \int_{\mathcal{M}} \operatorname{tr}\big(B \wedge F(A)\big). \tag{3}$$

The equations of motion are $F(A) = 0$ (flat connection) and $\mathrm{d}_A B = 0$; the theory has no local degrees of freedom. Discretizing on the two-complex $\mathcal{F}$, one assigns a group element (holonomy) $g_e \in G$ to each edge and integrates the constraint $F(A) = 0$ face by face. Expanding the group delta function on the holonomy $g_f = \prod_{e\subset\partial f} g_e$ around each face by the Peter–Weyl theorem,

$$\delta_G(g_f) = \sum_{j_f} \dim(V_{j_f})\,\chi_{j_f}(g_f), \tag{4}$$

and performing the edge integrals, one obtains a state sum in which each face carries an irreducible j_f, each edge carries an intertwiner produced by the Haar projection, and each vertex carries the recoupling invariant contracting the intertwiners on its edges. For $G = \mathrm{SU}(2)$ on a four-dimensional complex the vertex invariant is the $\{15j\}$ symbol, and the partition function is the Ooguri model,

$$\mathcal{Z}_{BF}(\mathcal{F}) = \sum_{\{j_f\}} \prod_f \dim(V_{j_f}) \prod_v \{15j\}(j_{f\ni v}). \tag{5}$$

Theorem 4.1 (Triangulation invariance of the topological state sum, [S])**.** *The three-dimensional* SU(2) *state sum* (4)–(5) *(Ponzano–Regge in dimension three, Ooguri in dimension four) is formally invariant under the Pachner moves relating any two triangulations of a fixed piecewise-linear manifold. After regularization by replacing* SU(2) *with the quantum group $U_q(\mathfrak{su}_2)$ at a root of unity (Turaev–Viro in dimension three, Crane–Yetter in dimension four), the resulting sum is finite and is a genuine piecewise-linear invariant of the manifold.*

Proof sketch. Pachner-move invariance reduces, via the Biedenharn–Elliott (pentagon) identity and the orthogonality of $6j$ (respectively $15j$) symbols, to algebraic identities among recoupling coefficients; these are the defining coherence relations of the representation category. The classical sums diverge because the trivial representation appears with infinite multiplicity in an unbounded sum over spins; the quantum-group truncation at level k makes the set of admissible labels finite, and Turaev–Viro (dimension three) and Crane–Yetter (dimension four) proved the truncated sum converges and is move-invariant, hence a topological invariant. □

Theorem 4.1 is the sharpest rigorous statement in the vicinity: a state sum of representation labels reconstructs a topological invariant of the manifold, exactly. It is also a warning. A theory that is fully triangulation invariant is topological — it has no local degrees of freedom and cannot be gravity. Gravity must break the topological invariance in a controlled way, keeping local curvature while retaining enough structure to have a continuum limit. That controlled breaking is the job of the simplicity constraint.

4.2 Simplicity constraints

General relativity in the Plebanski (or Palatini–Cartan) formulation is BF theory for $G = \mathrm{SO}(4)$ (Riemannian) or $\mathrm{SO}(3,1)$ (Lorentzian) supplemented by a constraint forcing the two-form B to come from a tetrad: $B^{IJ} = \pm(e \wedge e)^{IJ}$ or its dual. This *simplicity constraint* is what turns the topological BF theory into gravity with local degrees of freedom. In terms of the bivector B_f^{IJ} attached to a face (a discretized triangle), simplicity requires B_f to be a *simple* bivector, one of the form $u \wedge w$ for vectors u, w.

For $\mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2)$, the universal (double) cover of SO(4), a bivector splits into self-dual and anti-self-dual parts, carrying spins (j_f^+, j_f^-); the two SU(2) factors are exactly the self-dual and anti-self-dual sectors of the covered rotation group. Discrete simplicity, in its strong (Barrett–Crane) form, sets

$$j_f^+ = j_f^-, \tag{6}$$

the *balanced* or *simple* representations. Geometrically, (6) says the self-dual and anti-self-dual areas agree, which is the statement that the bivector is simple. The Barrett–Crane model imposes (6) strongly, as an operator equation annihilating states, and builds the vertex from the SO(4)-invariant contraction of balanced representations.

Remark 4.2 (Why strong imposition is too strong, [H]). Imposing all simplicity constraints strongly, as operator equations, over-constrains the intertwiner degrees of freedom: the Barrett–Crane vertex fixes the edge intertwiner uniquely (the Barrett–Crane intertwiner), removing the intertwiner labels that should survive as independent quantum numbers. The consequence, recognized around 2007, is that the boundary states of the Barrett–Crane model do not match the spin-network states of canonical loop quantum gravity, whose nodes carry a full intertwiner space. This mismatch motivated the weak imposition of the Engle–Pereira–Rovelli–Livine and Freidel–Krasnov models. The status here is [H]: the mismatch is a physical-adequacy judgment about boundary data, not a theorem that Barrett–Crane is mathematically inconsistent.

4.3 The Engle–Pereira–Rovelli–Livine vertex and the Y_γ map

The Engle–Pereira–Rovelli–Livine (EPRL) model imposes simplicity *weakly*, in the sense of the expectation value of a linear constraint, and in doing so incorporates the Barbero–Immirzi parameter γ. The construction rests on a map embedding SU(2) representations into $\mathrm{SL}(2,\mathbb{C})$ (Lorentzian) or Spin(4) (Riemannian) ones.

In the Lorentzian case the relevant $\mathrm{SL}(2,\mathbb{C})$ representations are the unitary principal series $\mathcal{H}_{(\rho,k)}$, labeled by $\rho \in \mathbb{R}$ and $k \in \frac{1}{2}\mathbb{Z}$. Each such representation decomposes under the SU(2) subgroup as $\mathcal{H}_{(\rho,k)} = \bigoplus_{j\geq k} V_j$. The linear simplicity constraint, imposed weakly, selects the lowest SU(2) component and ties the labels through γ:

$$\rho_f = \gamma\, j_f, \qquad k_f = j_f. \tag{7}$$

Definition 4.3 (The Y_γ map)**.** The Engle–Pereira–Rovelli–Livine map is the isometric embedding

$$Y_\gamma :\ V_j \ \hookrightarrow\ \mathcal{H}_{(\gamma j,\, j)}, \qquad Y_\gamma\, |j,m\rangle = |(\gamma j, j);\, j,m\rangle, \tag{8}$$

sending the spin-j SU(2) representation to the lowest ($j' = k = j$) SU(2)-isotypic component of the principal series $\mathcal{H}_{(\gamma j,j)}$. In the Riemannian model Y_γ sends V_j into $V_{j^+} \otimes V_{j^-}$ with $j^\pm = \frac{1}{2}|1 \pm \gamma|\, j$ (assuming $\gamma < 1$; the general case carries signs).

Definition 4.4 (EPRL vertex amplitude)**.** For a vertex v dual to a four-simplex, with the ten faces carrying spins j_f and the five edges $a = 1, \ldots, 5$ carrying SU(2) intertwiners ι_a, the EPRL vertex amplitude is

$$A_v^{\mathrm{EPRL}}\big(j_f, \iota_a\big) = \int_{\mathrm{SL}(2,\mathbb{C})^4} \prod_{a=1}^{4} \mathrm{d}g_a\ \mathrm{Tr}_{K_5}\Bigg(\bigotimes_{a=1}^{5} g_a \triangleright Y_\gamma \iota_a\Bigg), \tag{9}$$

where nominally one $\mathrm{SL}(2,\mathbb{C})$ element is associated to each of the five edges, but one group element (say $g_5 = \mathbf{1}$) is held fixed to make the noncompact $\mathrm{SL}(2,\mathbb{C})$ integral finite, leaving the four integrations over $\mathrm{SL}(2,\mathbb{C})^4$ shown. Here $g_a \triangleright Y_\gamma \iota_a$ denotes the group element g_a

acting, in the representations $(\gamma j_f, j_f)$ carried by the faces $f \ni a$, on the Y_γ-embedded edge intertwiner, and Tr_{K_5} contracts the five resulting invariant tensors along the ten faces of the vertex — the ten edges of the complete graph K_5 of Figure 1. The contraction is holistic and does not factorize into per-face matrix elements: an intertwiner $\iota_a \in \mathrm{Inv}\big(\bigotimes_{f\ni a} V_{j_f}\big)$ is an entangled invariant tensor, so a single face carries no gauge-invariant matrix element of its own; only the full K_5 trace is well defined. In the Riemannian model $\mathrm{SL}(2,\mathbb{C})$ is replaced by $\mathrm{Spin}(4)$ and the integrals are over a compact group, so no element need be fixed.

The vertex (9) is the recoupling invariant of the K_5 pattern of Figure 1: five edges, ten faces, contracted along the complete graph. In the Riemannian case it reduces to a γ-weighted $\{15j\}$ symbol. The Freidel–Krasnov model reaches an amplitude of the same form through a coherent-state (rather than projector) route; for $\gamma < 1$ the two agree, and we treat them together as the weak-simplicity vertex.

Remark 4.5 (What is chosen, [H]). Three things in (9) are physical choices, not derivations: the weak (rather than strong) imposition of simplicity; the identification (7) of the continuous $\mathrm{SL}(2,\mathbb{C})$ label ρ with γ times the discrete spin; and the face amplitude A_f, taken here as a power of $\dim V_{j_f}$. Different choices give different models with different divergence and continuum behavior. The literature motivates (7) by matching the loop-gravity area spectrum $A_f \propto \gamma\, j_f$, which is itself a semiclassical input. We tag the identification of (9) as "the" gravitational vertex with status [H].

5 Semiclassical asymptotics

The strongest evidence that the EPRL vertex is a gravity amplitude is its large-spin limit. Uniformly rescale all ten spins $j_f \mapsto \lambda j_f$ and let $\lambda \to \infty$.

Proposition 5.1 (Regge asymptotics of the EPRL four-simplex vertex, [S]). *Let the boundary data of a single four-simplex vertex be a coherent (Livine–Speziale) spin-network state peaked on the geometry of a nondegenerate Euclidean or Lorentzian four-simplex with triangle areas $A_f = \gamma\, j_f$ (in Planck units). Then as $\lambda \to \infty$ the rescaled vertex amplitude has the stationary-phase asymptotic*

$$A_v^{\mathrm{EPRL}}(\lambda j_f) \;\sim\; \frac{1}{\lambda^{12}}\Big[\, N_+\, e^{i\lambda\, S_{\mathrm{Regge}}(j_f)} + N_-\, e^{-i\lambda\, S_{\mathrm{Regge}}(j_f)} \Big] \;+\; (\textit{degenerate terms}), \tag{10}$$

where $S_{\mathrm{Regge}}(j_f) = \sum_{f\ni v} A_f\, \Theta_f$ is the (single-simplex) Regge action of the four-simplex, Θ_f the interior dihedral angle *at the triangle f (a single simplex has no deficit angle), and $N_\pm$ are λ-independent prefactors. The leading term is proportional to* $\cos\big(S_{\mathrm{Regge}} + \textit{const}\big)$.

Status and provenance. This is the theorem of Barrett, Dowdall, Fairbairn, Gomes, and Hellmann (Riemannian EPRL) and Barrett, Dowdall, Fairbairn, Hellmann, and Pereira (Lorentzian EPRL). The proof is a stationary-phase analysis: the vertex (9) is an integral over $\mathrm{SL}(2,\mathbb{C})$ (or $\mathrm{Spin}(4)$) of a product of matrix elements written, on coherent boundary data, as $e^{S(g)}$ with S an "action" on group variables; the critical points of S are in bijection with reconstructions of a geometric four-simplex from the boundary areas and normals, and the on-shell value of S at a nondegenerate critical point is $\pm i\, S_{\mathrm{Regge}}$. Standard stationary-phase then gives (10): the integration runs over four copies of the six-dimensional group $\mathrm{SL}(2,\mathbb{C})$ (or $\mathrm{Spin}(4)$), so the Hessian at a nondegenerate critical point is 24-dimensional, and the stationary-phase prefactor is $\lambda^{-24/2} = \lambda^{-12}$. The result is [S]: a proved asymptotic in a well-defined integral-asymptotics setting. □

Two honest caveats accompany Theorem 5.1, both [H]. First, the two terms of the cosine are the two parity-related geometric reconstructions of the four-simplex from the boundary areas and normals: in the Lorentzian model these are the two time orientations, so the amplitude does not distinguish a geometry from its time-reverse, and extracting a causal, single-exponential propagator requires additional input. The Riemannian model is more delicate still: its stationary-phase locus contains, besides the two geometric (Regge) critical points, additional degenerate "vector geometry" configurations that contribute non-Regge terms unless the boundary coherent state is chosen to suppress them, so (10) is the leading behavior only on suitably peaked geometric boundary data. Second, and more consequentially, (10) is the asymptotics of a *single* vertex on *fixed* boundary data. It does not by itself say what happens when one sums over the bulk spins of a many-vertex complex, and it is exactly there that the flatness problem appears (Section 7.2).

6 Worked example: the four-simplex amplitude

We make the smallest nontrivial spin foam explicit: the two-complex dual to a single four-simplex, which has one bulk vertex, five edges (dual to the five boundary tetrahedra), and ten faces (dual to the ten boundary triangles). Its boundary is the spin network on the pentagram graph K_5: five nodes (tetrahedra), ten links (triangles).

6.1 Boundary data and the amplitude

Label the ten faces by spins $j_{ab} = j_{ba}$ for $1 \leq a < b \leq 5$ (the triangle shared by tetrahedra a and b), and the five edges by intertwiners $\iota_a \in \mathrm{Inv}\big(\bigotimes_{b\neq a} V_{j_{ab}}\big)$, each a four-valent intertwiner. The vertex amplitude (9) contracts the five intertwiners along K_5. In the Riemannian, $\gamma \to 0$ or topological limit it degenerates to the $\{15j\}$ symbol

$$A_v(j_{ab}, \iota_a) = \{15j\}\big(\{j_{ab}\}; \{\iota_a\}\big), \tag{11}$$

the unique (up to normalization) SU(2)-invariant contraction of five four-valent intertwiners whose external legs are the ten j_{ab}. The full state sum for this one-vertex complex with fixed boundary is just the single vertex weighted by its face and edge amplitudes:

$$\mathcal{Z}(\mathcal{F}_{\Delta_4})[b] = \prod_{a<b} A_f(j_{ab}) \ \prod_{a} A_e(\iota_a)\ A_v(j_{ab}, \iota_a). \tag{12}$$

There is no bulk sum for a single four-simplex with fully specified boundary: every face and edge is a boundary object. The sum over bulk labels first appears when two or more four-simplices are glued and share an internal triangle or tetrahedron.

6.2 Equilateral boundary and the asymptotic angle

Take the symmetric boundary $j_{ab} = j$ for all ten faces, the quantum analog of an equilateral four-simplex. By Theorem 5.1, as $j \to \infty$,

$$A_v(\lambda j) \;\sim\; \frac{1}{\lambda^{12}}\, 2\,|N|\, \cos\Big(\lambda\, S_{\text{Regge}}(j) + \varphi\Big), \qquad S_{\text{Regge}}(j) = 10\,\gamma\, j\, \Theta, \tag{13}$$

where $\Theta = \arccos(1/4)$ is the interior dihedral angle of a regular Euclidean four-simplex (the angle between two of its tetrahedral faces), and the total is ten triangles times the area γj times that dihedral angle. The amplitude oscillates in j with a frequency set by the Regge action; the appearance of the regular four-simplex dihedral angle is the sense in which "the representation labels know the geometry."

6.3 A solvable topological shadow of the refinement invariant

The four-simplex amplitude (11) is not something we can sum in closed form over refinements; that is the open problem. To exhibit the *refinement invariant* exactly, we drop to the solvable case: a finite-group BF (Dijkgraaf–Witten) state sum, the abelian, discrete-group avatar of the topological theory of Section 4.1. Let G be a finite abelian group. On a closed oriented surface Σ triangulated with V vertices, E edges, and F triangles, assign a group element $h_e \in G$ to each edge and define

$$\mathcal{Z}_G(\Sigma) = \frac{1}{|G|^V} \sum_{\{h_e\}} \prod_f \big[\, \textstyle\sum_{e\subset\partial f} \pm h_e = 0\big], \tag{14}$$

where $[\cdot]$ is 1 if the oriented holonomy around the face is trivial and 0 otherwise, and the normalization $|G|^{-V}$ divides out the gauge redundancy $h_e \mapsto h_e + (\phi_{t(e)} - \phi_{s(e)})$. The following is a clean, fully [S] statement.

Proposition 6.1 (Exact refinement invariance in the solvable case, [S])**.** *For finite abelian G, the state sum (14) depends only on the topology of Σ:*

$$\mathcal{Z}_G(\Sigma) = |G|^{\,1-\chi(\Sigma)}, \tag{15}$$

with $\chi(\Sigma) = V - E + F$ the Euler characteristic. In particular it is invariant under any subdivision (refinement) of the triangulation: $\mathcal{Z}_G(S^2) = |G|^{-1}$ for every triangulation of the sphere, and $\mathcal{Z}_G(T^2) = |G|$ for every triangulation of the torus.

Proof. The face constraints say the coloring is a G-valued discrete flat connection, i.e. a cocycle in $Z^1(\Sigma; G)$; the number of them is $|Z^1|$. The gauge redundancy is B^0-worth of coboundaries, and for a connected Σ the map $C^0 \to Z^1$ has kernel the constants, so the count of gauge orbits is $|Z^1|/|G|^{V-1}$. Hence $\mathcal{Z}_G = |Z^1|/|G|^V$. For abelian G, $|Z^1| = |H^1| \cdot |B^1| = |G|^{2g} \cdot |G|^{V-1}$ on a genus-g surface (using $\dim B^1 = V-1$ for connected Σ and $|H^1| = 2g$-dimensional). The count $|H^1| = |G|^{2g}$ is the topological input: a flat G-connection up to gauge is a homomorphism $\pi_1(\Sigma_g) \to G$, and since $\pi_1(\Sigma_g)$ has $2g$ generators subject to the single relation $[a_1, b_1]\cdots[a_g, b_g] = 1$, which is automatically satisfied once G is abelian, one has $|H^1| = |\operatorname{Hom}(\pi_1(\Sigma_g), G)| = |G|^{2g}$. Therefore $\mathcal{Z}_G = |G|^{2g+V-1}/|G|^V = |G|^{2g-1} = |G|^{1-\chi}$, since $\chi = 2-2g$. Because χ is a topological invariant, the value is unchanged under refinement. $\square$

Theorem 6.1 is the exactly-solvable shadow of what the gravitational state sum is being asked to do: produce an amplitude that stabilizes under refinement. In the topological case it stabilizes perfectly, because the theory has no local degrees of freedom. Gravity must stabilize *while* carrying local curvature, and whether it can is open. The companion Haskell code (Section 9, Section A) computes (14) by brute force for $G = \mathbb{Z}_n$ on several triangulations and checks (15) and refinement invariance, alongside the SU(2) recoupling bookkeeping that supports (11).

7 Refinement, the continuum limit, and the flatness problem

7.1 Refinement and cylindrical consistency

A single two-complex is a truncation of the degrees of freedom, like a single lattice in lattice gauge theory. Because gravity is background independent, there is no fixed lattice spacing to send to zero; refinement means subdividing the complex, and the continuum theory (if it exists) is a limit over the directed set of refinements.

Definition 7.1 (Refinement and cylindrical consistency)**.** A *refinement* $\mathcal{F} \to \mathcal{F}'$ is a subdivision: each cell of $\mathcal{F}$ is partitioned into cells of $\mathcal{F}'$, inducing an embedding of boundary graphs $\Gamma \hookrightarrow \Gamma'$ and hence an isometry $\iota_{\Gamma\Gamma'} : \mathcal{H}_\Gamma \hookrightarrow \mathcal{H}_{\Gamma'}$ of boundary Hilbert spaces. The family of amplitudes $\{\mathcal{Z}(\mathcal{F})\}$ is *cylindrically consistent* if for every refinement,

$$\mathcal{Z}(\mathcal{F}) = \iota^\dagger_{\Gamma\Gamma'}\, \mathcal{Z}(\mathcal{F}')\, \iota_{\Gamma\Gamma'}, \tag{16}$$

i.e. coarser amplitudes are restrictions of finer ones. When (16) holds, the amplitudes define a single map on the projective limit $\mathcal{H}_{\Gamma_\infty} = \varinjlim \mathcal{H}_\Gamma$, a genuine continuum amplitude.

Remark 7.2 (The continuum limit is open, [H]). Generic spin-foam models are *not* cylindrically consistent as written: refining the complex changes the amplitude. The proposed remedies are (i) to construct improved, consistent amplitudes as fixed points of a coarse-graining (renormalization group) flow on the space of amplitudes, following Dittrich and collaborators, or (ii) to sum over all two-complexes, e.g. via a group field theory whose Feynman expansion generates spin foams. Neither program has produced a proof that a continuum limit exists and reproduces general relativity. We state this as the primary open problem of the regime and tag every claim about the existence of the limit [H].

7.2 The flatness problem

The most concrete obstruction to a naive continuum limit is the flatness (or "cosine," or "accidental curvature") problem. Consider a complex with internal faces and take the large-spin limit while *summing* over the bulk spins. On coherent boundary data each vertex contributes $e^{\pm iS^{(v)}}$ by Theorem 5.1, and the bulk amplitude becomes a stationary-phase integral over the group elements g_e on the internal edges (together with the bulk spin sums). The critical-point equations of these group integrals are the geometric content of the flatness problem: at each internal face f they force the product of edge holonomies around f to be trivial,

$$H_f = \prod_{e \subset \partial f} g_e = \mathbf{1}, \tag{17}$$

and a trivial holonomy around every internal hinge is exactly the statement that the reconstructed geometry has vanishing *deficit angle* there, $\varepsilon_f = 2\pi - \sum_{v \supset f} \theta_f^{(v)} = 0$, with $\theta_f^{(v)}$ the interior dihedral angle contributed by the simplex at vertex v. That is, the dominant configurations in the summed large-spin limit are flat. It is the group-integral saddle (17), not the variation of the phase with respect to the areas $A_f = \gamma j_f$, that carries the obstruction: for a generic Barbero–Immirzi parameter the naive area-variation condition holds only modulo 2π and does not by itself force $\varepsilon_f = 0$, whereas the holonomy condition does. Curved geometries, which carry nonzero deficit angles and are exactly the physical content of general relativity, are therefore suppressed. This was flagged by Bonzom and given a partition-function-level (wavefront-set) derivation by Hellmann and Kaminski, and confirmed numerically in EPRL and BF cases by Donà and collaborators.

Remark 7.3 (Status of the flatness problem, [H]). The precise scope of (17) is contested: whether the flatness holds for all refinements or is an artifact of particular limits and boundary conditions, and whether it survives at finite spin, remain under active investigation. What is not contested is that it is a genuine obstruction to the simplest hoped-for scenario in which refining a spin foam and taking large spins directly yields curved classical gravity. The status is [H]: it is an asymptotic/numerical property of the models, sharply posed, and unresolved.

7.3 Effective spin foams

A recent reframing (Asante, Dittrich, Haggard) confronts the flatness problem by building an *effective* spin foam directly from the discrete gravity (Area–Regge) action, taking the quantum area spectrum as fundamental and imposing the gluing (shape-matching) constraints as strongly as the uncertainty relations of the area variables allow. In these models curved Regge geometries are not suppressed; the flat dominance of (17) is traced to over-imposing constraints that the area uncertainties forbid imposing sharply. This is real progress on the flatness problem, and it clarifies which choices in Theorem 4.5 drive it. It is not a resolution of the continuum-limit problem: effective spin foams are still defined on a fixed complex, and the limit over refinements remains open. We tag the effective-spin-foam resolution of flatness [H] and the continuum limit itself open.

7.4 Divergences and the sum over complexes

Two further honest points. First, at fixed complex the sum over bulk spins can diverge: internal "bubbles" (closed surfaces in the complex) produce factors resembling $\sum_j (\dim V_j)^p$ that diverge for the natural face amplitudes, the analog of loop divergences in lattice gauge theory. The quantum-group truncation of Theorem 4.1 regulates the topological case; for gravity the divergence structure depends on the face amplitude and is part of what a renormalization treatment must control. Second, summing over two-complexes (rather than refining a fixed one) is the group-field-theory strategy; its convergence and its independence of the chosen family of complexes are not established. We tag the sum over complexes [H/P]: heuristic where a concrete group field theory is specified, speculative as a claimed definition of the full nonperturbative theory.

8 Relation to the other regimes and the S/H/P calculus

The library treats twelve quantum-gravity regimes as a modular site, composed by the weakest-link calculus of Theorem 2.1 rather than unified. We locate spin foams among them.

8.1 Spin networks as boundary data (QG-III)

The most direct relation is internal to module QG-III. The boundary Hilbert spaces $\mathcal{H}_\Gamma$ on which spin-foam amplitudes act *are* the spin-network states of the companion regime: graphs with SU(2) edge labels and intertwiner vertex labels. The kinematics is settled (the kinematical Hilbert space and area/volume spectra are [S]; the representation is unique by the Lewandowski–Okołów–Sahlmann–Thiemann theorem). The dynamics — the spin-foam amplitude between these states — is the bare-[H] link. The composite "spin-network kinematics $*$ spin-foam dynamics" is therefore

$$[\mathsf{S/H}] \; * \; [\mathsf{H}] \; = \; [\mathsf{H}],$$

the precise statement that a complete covariant quantum-gravity dynamics inherits the heuristic status of its dynamical half. This is not pessimism; it is bookkeeping. The kinematics really is settled, and the calculus records exactly that the settled part does not settle the whole.

8.2 Classical geometry as the target (QG-I)

Module QG-I — Lorentzian manifolds and the moduli of Einstein solutions — is the target that the missing continuum limit must reproduce. The one rigorous rung of the bridge is the semiclassical asymptotics of Theorem 5.1: a single vertex, on Regge boundary data, reproduces $e^{\pm i S_{\mathrm{Regge}}}$, the discrete Einstein–Hilbert action. Extending this from one vertex on fixed data to a refined complex with summed bulk, and thence to smooth solutions, is exactly the continuum-limit problem, obstructed by flatness. The composite

$$(\text{classical geometry}, [\mathsf{S/H}]) \; * \; (\text{spin foam}, [\mathsf{H}]) = [\mathsf{H}],$$

and it is optimistic even at [H]: no theorem currently establishes that QG-I is the continuum limit of the state sum.

8.3 Asymptotic safety and renormalization (QG-VII)

The coarse-graining program for spin foams (Theorem 7.2) is a renormalization group flow on amplitudes, and it is natural to ask whether a fixed point of that flow could coincide with the non-Gaussian fixed point of asymptotic safety (module QG-VII). Both are bare-[H] entries, so any composite is bounded by [H]; but here the weakest-link bound is an upper bound only. No rigorous bridge exists between a combinatorial refinement fixed point and a continuum theory-space fixed point, so the [H] of the composite should be read as "neither endpoint exceeds heuristic, hence no chain built from them can," not as "the bridge is established." The recent tensor-network and Monte Carlo methods make the refinement fixed-point question computationally accessible, which is where a genuine comparison might eventually be made.

8.4 Gauge, BF, and cobordism (QG-II, TQFT)

The route from BF theory to gravity (Section 4) ties this regime to the Palatini–Cartan and Plebanski formulations of classical gravity (module QG-I) and, through the constraint structure, to the gauge/BV–BRST regime (module QG-II): simplicity is a second-class constraint whose proper treatment is a homological question. The gluing law (Theorem 3.5) places spin foams in the cobordism/TQFT column: a spin foam is a morphism between spin networks. The composite with the ([S/H]) cobordism entry stays [S/H] for the topological BF models and degrades to [H] for the gravitational ones, whose triangulation dependence breaks the strict TQFT axioms.

8.5 Explicit non-identifications

To keep the modularity honest we record what spin foams are *not*. A spin foam is not a spacetime: it is a combinatorial object whose relation to a smooth four-manifold is, beyond the topological/asymptotic level, exactly the open problem. The state sum is not a proven path integral for quantum gravity: it is a proposal whose vertex is chosen, not derived. And the sum over refinements is not known to converge: cylindrical consistency fails generically, and the continuum theory is a program, not a theorem. The weakest-link calculus exists to prevent these non-identifications from being quietly elided.

9 A Haskell model of the state sum

To make the reconstruction concrete and checkable we accompany the paper with a small Haskell development, split into base-only modules (association lists, no external containers) so that it compiles with a bare `ghc`. It models the two layers of the paper that are exactly computable: the SU(2) recoupling bookkeeping underlying the vertex amplitude, and the finite-group BF (Dijkgraaf–Witten) state sum that exhibits the refinement invariant exactly.

The recoupling layer (`Core`) implements SU(2) spins as half-integers, irreducible dimensions $2j+1$, the triangle admissibility condition, Clebsch–Gordan fusion, and the dimension of an intertwiner space $\mathrm{Inv}(\bigotimes_i V_{j_i})$ as the multiplicity of the trivial representation, by two independent routes (iterated fusion and, for four legs, the recoupling count). A checkable property asserts the two routes agree, and that a four-simplex boundary labeled by admissible spins has the expected nonzero intertwiner support — the data on which the $\{15j\}$ vertex (11) is defined.

The state-sum layer (`StateSum`, `Complexes`) implements the amplitude (14) for $G = \mathbb{Z}_n$ by brute-force enumeration of edge colorings with the face-holonomy constraints, on three explicit triangulations: the boundary of a tetrahedron and an octahedron (two triangulations of S^2) and a minimal two-triangle torus. A checkable property asserts Theorem 6.1: that both sphere triangulations give $\mathcal{Z} = |G|^{-1}$ (refinement invariance) and that the torus gives $\mathcal{Z} = |G|$, matching $|G|^{1-\chi}$. This is the exact, solvable version of the paper's core invariant — the refinement behavior of a state-sum amplitude — in the one case where it is fully under control.

The `Main` module runs the demonstrations and the property suite and exits with a nonzero status if any property fails. The core module is reproduced in Section A; the full sources (`Core`, `StateSum`, `Complexes`, `Properties`, `Main`) accompany the submission as ancillary files.

10 Limitations and open problems

We collect the honest status.

Continuum/refinement limit (open, [H]). The central open problem: whether the family $\{\mathcal{Z}(\mathcal{F})\}$ over refinements is cylindrically consistent, or admits a renormalization-group fixed point, defining a continuum amplitude that reproduces general relativity. Coarse-graining/tensor-network and group-field-theory programs exist but have not closed this. No theorem asserts that module QG-I is the continuum limit of the state sum.

The flatness problem (open, [H]). The naive summed large-spin limit is dominated by flat configurations (17). Effective spin foams reframe the amplitude so that curved configurations survive, which is progress on this specific obstruction, but not a resolution of the continuum limit.

Choice of vertex and measure ([H]). The EPRL/FK vertex is fixed by a chosen (weak) imposition of simplicity, a chosen label identification (7), and a chosen face amplitude. Different choices give different models; there is no derivation from first principles selecting one.

Divergences ([H]). At fixed complex the bulk sum can diverge through internal bubbles; the divergence structure depends on the face amplitude and is part of the renormalization problem.

Lorentzian signature and causality ([H]). The cosine in (10) contains both time orientations; extracting a causal, single-exponential propagator, and controlling the noncompact $\mathrm{SL}(2,\mathbb{C})$ integrals, are technical and conceptual issues beyond the scope of the settled asymptotics.

Sum over topologies ([H/P]). Whether and how to sum over the topology of the underlying manifold, and whether the group-field-theory generating function defines the nonperturbative theory, is speculative.

11 Discussion

The reconstruction thesis is sharp in this regime, and so is its limit. What is reconstructed, rigorously, is a family of amplitudes: a colored two-complex, through its state sum, defines a well-defined map between spin-network boundary Hilbert spaces (Theorem 3.4), the map composes by gluing (Theorem 3.5), the topological version is an exact manifold invariant (Theorem 4.1), and a single gravitational vertex reproduces the discrete Einstein–Hilbert action in its large-spin asymptotics (Theorem 5.1). At this level the slogan "a spacetime history is a pattern of representation labels on a two-complex, and its amplitude is a product of recoupling invariants" is not a metaphor: there is no metric in the data.

What the thesis has not earned, in this regime, is the continuum. A smooth curved spacetime with its Einstein dynamics is supposed to emerge as the two-complex is refined, and it is precisely there that the construction is unfinished: cylindrical consistency fails generically, the naive large-spin limit is flat, and the renormalization and group-field-theory programs that would fix this are open. The value of the status calculus is that it holds these two facts apart. The settled part is genuinely settled, and the calculus forbids it from lending its credibility to the unsettled continuum limit: a chain is as strong as its weakest warrant, and the weakest warrant here is bare [H].

Seen this way, spin foams are the covariant-dynamics counterpart of the spin-network kinematics: the kinematics gives the boundary states, the foams give the amplitudes between them, and together they would be a background-independent path integral for gravity — if the continuum limit existed. The neighboring regimes supply the rest of the picture: classical Lorentzian geometry is the target, asymptotic safety offers a continuum fixed point to compare against, and the BF/cobordism structure supplies the categorical grammar in which a history is a morphism.

12 Conclusion

We have presented covariant quantum geometry as a state sum on a colored two-complex and organized the regime around one invariant: the boundary amplitude and its refinement behavior. The proved statements — well-definedness of the truncated amplitude (Theorem 3.4), the topological state-sum invariant (Theorem 4.1), the gluing/composition law (Theorem 3.5), the exact refinement invariance of the solvable finite-group model (Theorem 6.1), and the Regge asymptotics of the EPRL four-simplex vertex (Theorem 5.1) — are [S]: standard mathematics in well-defined settings. The physical readings — that the EPRL/FK vertex is the correct gravitational dynamics, and above all that the family of amplitudes has a continuum limit reproducing general relativity — are [H], and the continuum limit is openly unresolved, obstructed at the simplest level by the flatness problem.

The regime composes modularly with its neighbors through the weakest-link calculus, which bounds every composite by its bare-[H] spin-foam link, and the accompanying Haskell model turns the exactly-solvable part of the invariant — refinement invariance of a topological state sum — into a machine-checkable property. The honest headline is narrow and firm: a spin foam is a well-defined amplitude between representation-theoretic boundary states, and its single-vertex semiclassical limit is the Regge action; both are theorems. The broad headline — that this is a continuum theory of quantum gravity — remains a hypothesis, and we have been careful to say so.

A The core Haskell module

For reproducibility we reproduce the recoupling core, which models SU(2) irreducibles, Clebsch–Gordan fusion, and intertwiner-space dimensions by two independent routes. It uses only `base` (association lists), compiles with a bare `ghc`, and underlies the checkable properties of Section 9. The remaining modules (`StateSum`, `Complexes`, `Properties`, `Main`) accompany the submission as ancillary files.

```
{-# LANGUAGE ScopedTypeVariables #-}
-- Module Core: SU(2) recoupling bookkeeping for spin-foam vertices.
-- All content here is the [S] (standard mathematical) layer.
module Core
  ( Spin, spin, halfInt, dimIrrep, casimir, triangle
  , fuse, fuseMany, intertwinerDim, intertwinerDim4
  ) where

-- A spin j is a nonnegative half-integer stored as twice its value (an Int),
-- so j = 1/2 is HalfInt 1, j = 1 is HalfInt 2. Base-only, no Data.Ratio needed.
newtype Spin = Spin Int deriving (Eq, Ord)   -- Spin k represents j = k/2

instance Show Spin where
  show (Spin k)
    | even k    = show (k `div` 2)
    | otherwise = show k ++ "/2"

-- Smart constructor from twice-the-spin (rejects negatives).
halfInt :: Int -> Spin
halfInt k | k < 0     = error ("halfInt: negative " ++ show k)
          | otherwise = Spin k

-- Convenience: integer spin.
spin :: Int -> Spin
spin j = halfInt (2 * j)

-- Dimension of the carrier V_j: 2j + 1 = k + 1.
dimIrrep :: Spin -> Int
dimIrrep (Spin k) = k + 1

-- Casimir j(j+1), returned as twice-numerator over 4 to stay in Int-space:
```

```
-- j(j+1) = (k/2)(k/2 + 1) = k(k+2)/4.
casimir :: Spin -> Rational'
casimir (Spin k) = R (k * (k + 2)) 4

-- A tiny rational to avoid Data.Ratio (containers/ratio kept base-simple).
data Rational' = R Int Int
instance Show Rational' where show (R a b) = show a ++ "/" ++ show b

-- Triangle admissibility for j1 (x) j2 -> j3: |j1-j2| <= j3 <= j1+j2 and
-- j1 + j2 + j3 integral (i.e. k1 + k2 + k3 even).
triangle :: Spin -> Spin -> Spin -> Bool
triangle (Spin a) (Spin b) (Spin c) =
  abs (a - b) <= c && c <= a + b && even (a + b + c)

-- Clebsch-Gordan fusion V_a (x) V_b = sum_{k=|a-b|}^{a+b} V_k, step 1 in j
-- (step 2 in the twice-spin integer). Returned as (spin, multiplicity) alist.
fuse :: Spin -> Spin -> [(Spin, Int)]
fuse (Spin a) (Spin b) =
  [ (Spin k, 1) | k <- [abs (a - b), abs (a - b) + 2 .. a + b] ]

-- Iterated fusion of a list of irreducibles, accumulating multiplicities in
-- a base-only association list.
fuseMany :: [Spin] -> [(Spin, Int)]
fuseMany []       = [(spin 0, 1)]
fuseMany (x : xs) = foldl step [(x, 1)] xs
  where
    step acc j = collect
      [ (k, m * mult) | (a, m) <- acc, (k, mult) <- fuse a j ]

-- Sum multiplicities of equal keys in an association list.
collect :: [(Spin, Int)] -> [(Spin, Int)]
collect = foldr ins []
  where ins (k, m) [] = [(k, m)]
        ins (k, m) ((k', m') : rest)
          | k == k'   = (k', m + m') : rest
          | otherwise = (k', m') : ins (k, m) rest

-- Intertwiner-space dimension Inv(V_{j1} (x) ... (x) V_{jn}) = multiplicity
-- of the trivial rep (spin 0) in the full tensor product.
intertwinerDim :: [Spin] -> Int
intertwinerDim js =
  maybe 0 id (lookup (spin 0) (fuseMany js))

-- Independent four-valent count via the recoupling channel:
-- #{ k : max(|j1-j2|,|j3-j4|) <= k <= min(j1+j2,j3+j4), parities match }.
intertwinerDim4 :: Spin -> Spin -> Spin -> Spin -> Int
intertwinerDim4 (Spin a) (Spin b) (Spin c) (Spin d) =
```

```
length
  [ () | k <- [lo, lo + 2 .. hi]
       , even (a + b + k), even (c + d + k) ]
where lo = max (abs (a - b)) (abs (c - d))
      hi = min (a + b) (c + d)
```

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