Amplitudes as Canonical Forms: Positive Geometries, Residue Trees, and the Conjectural Bridge to Motivic Coactions as the Amplitude Layer of Reconstructed Spacetime

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Abstract

We treat perturbative scattering amplitudes not as objects derived from a spacetime Lagrangian, a Feynman expansion, or a path integral, but as canonical forms of positive geometries: top-degree rational differential forms determined uniquely by the requirement that their singularities are simple logarithmic poles supported exactly on the boundary strata of a semialgebraic region, with unit residue recursively. In this reading, locality and unitarity—the poles of an amplitude and its factorization on those poles—are consequences of the boundary combinatorics of the geometry rather than separately imposed physical axioms. The core invariant of the regime is therefore the recursive residue/boundary stratification, which we call the *residue tree*.

We give a self-contained account of the positive-geometry framework of Arkani-Hamed, Bai and Lam, the canonical-form recursion, and its realization in kinematic space by the Arkani-Hamed–Bai–He–Yan (ABHY) associahedron, whose canonical form is the tree-level bi-adjoint scalar amplitude. Our results are stated as labelled propositions and each carries the project's S/H/P epistemic tag (standard / heuristic / speculative). The existence and uniqueness of canonical forms, the residue recursion, and its identification with amplitude factorization are standard mathematics [S]; the identification of the canonical form with a physical S-matrix is heuristic and holds only for special theories [S/H]; and the central conjecture of the field—that the residue tree of the geometry coincides with the motivic coaction tree of the associated period—is heuristic in the verified low-weight, mixed-Tate cases and speculative in general [H/P]. We work the five-point (pentagon) associahedron in full: its explicit canonical form, an explicit residue computation exhibiting factorization, and a careful, honestly-labelled comparison of its residue tree with the coaction tree of the transcendental (stringy / twisted) completion, using the coaction of Abreu–Britto–Duhr–Gardi–Matthew where both integrand and contour are positive geometries. We locate the regime among its neighbours: the double copy expressed through the inverse of the bi-adjoint (associahedron) amplitude, the motivic coaction machinery, and the celestial change of basis, tracking the epistemic status under the worst-component-wins composition calculus. We close with the honest obstructions: elliptic and non-Tate amplitudes that break the tame mixed-Tate coalgebra, and the absence of a positive geometry for a general gravitational S-matrix beyond double-copy and toy-cosmological constructions. A companion Haskell development computes associahedron canonical forms, extracts residues, verifies factorization on the pentagon, and checks the combinatorial invariants exactly.

Contents

1 Introduction

1.1 The reconstruction thesis in the amplitude regime

The governing perspective of this project is that quantum gravity is not, in the first instance, the quantization of material objects placed in a pre-existing spacetime. It is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures. Different regimes of the theory instantiate this idea with different invariants. The present paper concerns the regime of perturbative scattering amplitudes, where the invariant is unusually sharp and the reconstruction unusually complete.

The conventional route to an amplitude begins with a Lagrangian, expands the path integral into Feynman diagrams, and sums. Poles arise because internal propagators can go on shell; the residues at those poles factorize into products of lower-point amplitudes because the theory is unitary; and the whole construction respects locality because the Lagrangian is built from local fields. In this telling, locality and unitarity are inputs, and the analytic structure of the amplitude is an output that one verifies after the fact.

The positive-geometry program of Arkani-Hamed and collaborators inverts this order of explanation [1, 2, 3]. One begins with a geometric object: a region $X_{\geq 0}$ inside a real projective (or Grassmannian) variety, carved out by inequalities, whose boundary decomposes recursively into lower-dimensional regions of the same type. To such an object one attaches a unique rational top-form Ω, its *canonical form*, singled out by the demand that its poles be simple, sit precisely on the boundary components of $X_{\geq 0}$, and reproduce recursively—on each boundary—the canonical form of that boundary. The claim, established in specific theories, is that this Ω *is* the amplitude. No Lagrangian, no diagram sum, and no path integral enter the definition of Ω. Locality (the location of the poles) and unitarity (the factorization on them) are read off from the boundary combinatorics of $X_{\geq 0}$. They are theorems about a polytope, not axioms about a field theory.

This is, across the twelve regimes catalogued in the project's knowledge base, the single most complete example of an observable reconstructed directly from a geometric/representation-theoretic invariant. In the theories where the correspondence is established, the equation "amplitude = canonical form" holds without remainder. That is a strong statement, and we are careful throughout to say exactly where it holds and where it does not.

1.2 The core invariant: the residue tree

Fix a positive geometry $(X, X_{\geq 0}, \Omega_X)$ of dimension d. Its boundary $\partial X_{\geq 0}$ is a union of codimension-one positive geometries C_i; each C_i has its own boundary, a union of codimension-two pieces; and so on down to points. The resulting rooted tree— X at the root, its facets as children, their facets as grandchildren, terminating at 0-dimensional strata—is what we call the *residue tree* of X, written $\mathsf{R}(X)$. It is the same data as the face poset of $X_{\geq 0}$, organized by the residue recursion. Physically it is the complete catalogue of factorization channels of the amplitude: each root-to-leaf path is a maximal sequence of compatible on-shell conditions, i.e. a Feynman-like tree, and each internal node is a factorization onto a product of sub-amplitudes.

The residue tree is the invariant this paper takes as primary. Every claim we make is, in the end, a claim about $\mathsf{R}(X)$: that it exists and is finite; that the canonical form is reconstructed from it; that its leaves are the vertices of the polytope and its root-to-leaf paths are the cubic trees of the theory; and—conjecturally—that it is isomorphic to a second tree of entirely different origin, the coaction tree of the amplitude's period.

1.3 The conjectural bridge: residue tree = coaction tree

There is a parallel decomposition of amplitudes that comes not from geometry but from number theory. A large class of amplitudes, once integrated, evaluate to periods: multiple polylogarithms and multiple zeta values in the mixed-Tate case. Such periods carry a motivic coaction Δ, a coproduct-like map that peels off, layer by layer, the transcendental content of the function [12, 14, 15, 17]. Iterating Δ produces a rooted tree—the *coaction tree*—whose nodes are the successive "pieces" of the period and whose leaves are the primitive weight-one logarithms.

The conjecture that organizes this regime, item 1 of the project's open-problem queue, is that these two trees are the same:

$$\mathsf{R}(X) \;\cong\; \mathsf{C}(\Omega_X),$$

the geometric residue tree of the positive geometry is isomorphic to the motivic coaction tree of its canonical form (suitably completed to a transcendental function). The statement is verified in special low-multiplicity, low-weight, mixed-Tate cases and is supported structurally by the observation that cluster coordinates on the Grassmannian simultaneously index the boundary strata of the geometry and the letters of the polylogarithm [13]. The strongest concrete bridge to date is the coaction of Abreu–Britto–Duhr–Gardi–Matthew, built on twisted (co)homology, in which both the integrand and the integration contour are positive geometries and the coaction's contour factor runs literally over the boundary strata of the contour [18]. We take that construction as the technical heart of our worked comparison in Section 6.

1.4 Epistemic discipline

We use the project's three-valued status calculus. A warrant is S (standard mathematics or established mathematical physics), H (strong heuristic physical representation), or P

(speculative ontological extension), with $\mathsf{S} < \mathsf{H} < \mathsf{P}$. Composite labels such as $\mathsf{S/H}$ record a claim that is mathematically standard but heuristic as physics. Composition is monotone and worst-component-wins: chaining regimes can never raise the composite status above its weakest link. Every displayed result carries a tag in brackets, e.g. [S], [S/H], [H], [H/P], [P]. This is not decoration. The uniqueness of canonical forms is mathematics [S]; that a given canonical form computes a physical S-matrix is a theorem only for special theories and is otherwise a hope [S/H]; and the residue-tree/ coaction-tree identification is, in full generality, a conjecture we do not know how to prove [H/P]. Keeping these apart is the point of the exercise.

1.5 Organization

Section 2 recalls the relevant rows of the project dictionary and the status calculus. Section 3 develops the positive-geometry framework: canonical forms, uniqueness, the residue recursion, and the residue tree as a functor. Section 4 realizes the framework in kinematic space through the ABHY associahedron, and reviews the amplituhedron and cosmological polytopes. Section 5 states the main results. Section 6 works the pentagon in full, including the honest residue-tree/ coaction-tree comparison. Section 7 places the regime among its neighbours (double copy, motivic coactions, celestial holography) and tracks the composed status. Section 8 is a candid account of the obstructions and open problems. Section 9 concludes. Section A documents the companion code.

2 The dictionary and the status calculus

We reproduce the rows of the project's representation dictionary that this paper instantiates, and fix the status calculus used throughout.

Definition 2.1 (Representation entry, [S])**.** A *representation entry* is a tuple $E = (M, P, \tau, \sigma)$ consisting of a mathematical object M, a physical target P, a translation datum $\tau : M \rightsquigarrow P$, and an epistemic status $\sigma \in \{\mathsf{S}, \mathsf{H}, \mathsf{P}, \dots\}$ (a nonempty tuple of warrants, rendered as a composite label). The regime QG-IV of this project is populated by the entries in Table 1.

Status	Math M	Physical target P	Invariant
S/H	Positive geometry with canonical form	Amplitude from boundary/residue data	residues, boundaries
S/H	Color–kinematics dual numerators	Gravity amplitude as gauge^2	double-copy numerators
H	Motivic coaction / Goncharov Lie coalgebra	Decomposition anatomy of the amplitude	coaction, cobracket
H	Celestial conformal data	Flat-space S-matrix as celestial correlator	soft/asymptotic Ward id.

Table 1: The QG-IV dictionary rows instantiated by this paper. Statuses are the entries' isolated warrants; composite claims degrade by the worst component.

Definition 2.2 (Status calculus, [S]). Order the warrants $\mathsf{S} < \mathsf{H} < \mathsf{P}$. For statuses σ, σ' (tuples of warrants) define the composite $\sigma \star \sigma'$ to be the status whose worst component is $\max(\text{worst}\,\sigma, \text{worst}\,\sigma')$; more precisely $\star$ is the componentwise max on the lattice of warrants. Then $\star$ is associative and commutative, S is a two-sided unit ($\mathsf{S} \star \sigma = \sigma$), and $\star$ is monotone: $\text{worst}(\sigma \star \sigma') \geq \text{worst}(\sigma)$.

Remark 2.3. Theorem 2.2 is the formal engine of Section 7. When we compose the positive-geometry entry S/H with the motivic-coaction entry H to make the residue-tree/coaction-tree bridge, the result is at best H, and as soon as the general (non-Tate) case is invoked it degrades to P. No amount of chaining upgrades a heuristic to a theorem. The Haskell companion (Section A) implements $\star$ and checks the unit and monotonicity laws exactly on the finite warrant lattice.

3 The positive-geometry framework

We follow Arkani-Hamed, Bai and Lam [1], with the Hodge-theoretic refinements of Brown and Dupont [7] and the expository framing of Lam [5] and Herrmann–Trnka [6].

3.1 Positive geometries and canonical forms

Definition 3.1 (Positive geometry, [S]). A *positive geometry* of dimension d is a triple $(X,\, X_{\geq 0},\, \Omega_X)$ where X is a d-dimensional irreducible complex projective variety defined over $\mathbb{R}$, $X_{\geq 0} \subseteq X(\mathbb{R})$ is a closed oriented semialgebraic subset whose Zariski closure is X, and Ω_X is a rational d-form on X (the *canonical form*), subject to the following recursive conditions. If $d = 0$, then X is a point and $\Omega_X = \pm 1$ (an orientation). If $d > 0$, then $\partial X_{\geq 0} = \bigcup_i C_i$ where each $(C_i, C_{i,\geq 0})$ is (the closure of) a $(d-1)$-dimensional positive geometry, and Ω_X has poles only along the C_i, all simple, with

$$\operatorname{Res}_{C_i} \Omega_X \;=\; \Omega_{C_i}, \tag{1}$$

the canonical form of the boundary component C_i, for every i; Ω_X has no other singularities.

The residue in (1) is the Poincaré residue: if in local coordinates $C_i = \{x_1 = 0\}$ and $\Omega_X = \frac{dx_1}{x_1} \wedge \eta + \cdots$ with η regular near C_i and the ellipsis regular, then $\operatorname{Res}_{C_i} \Omega_X = \eta|_{x_1=0}$.

Theorem 3.2 (Uniqueness of canonical forms, [S]; [1, Thm. 2.1]). *If a canonical form satisfying Theorem 3.1 exists for* $(X,\, X_{\geq 0},\, \Omega_X)$*, it is unique.*

Proof sketch. Suppose Ω, Ω' both satisfy the axioms. Their difference $\Omega - \Omega'$ is a rational d-form. On each boundary component C_i the residues agree by (1) and induction on dimension (the base case $d = 0$ being the fixed orientation), so $\Omega - \Omega'$ has at worst simple poles with vanishing residues along each C_i; a simple pole with zero residue is no pole. Hence $\Omega - \Omega'$ is a global regular d-form on the projective variety X. For the rational varieties that arise (and, in the Brown–Dupont framework, for a genus-zero pair) the space of global holomorphic top-forms is trivial for the relevant weight reasons, forcing $\Omega - \Omega' = 0$. The clean statement is that a top-form with only simple poles, prescribed unit residues on a normal crossings boundary, and no other poles, is determined by its residues. ∎

Remark 3.3 (Existence, [S]). Existence is more delicate than uniqueness and is not automatic. Brown and Dupont [7] give a cohomological existence criterion: the canonical form is the framing of a "genus-zero pair" (X, Y) of varieties (with Y the boundary divisor), existing precisely when the top weight-graded piece of the relative cohomology $H^d(X \setminus Y, —)$ is one-dimensional and of Tate type. For polytopes and hyperplane arrangements existence is classical.

3.2 Polytopes and the simplex

The simplest positive geometries are convex polytopes in projective space, and they suffice for the associahedron. We record the two facts we need.

Proposition 3.4 (Canonical form of a simplex, [S]). *Let $\Delta \subset \mathbb{P}^d$ be the simplex cut out by $d+1$ inequalities $\ell_a(x) \geq 0$, $a = 0, \ldots, d$, with ℓ_a linear in homogeneous coordinates. In an affine chart its canonical form is*

$$\Omega_\Delta \;=\; \pm\, \mathrm{dlog}\,\frac{\ell_1}{\ell_0} \wedge \mathrm{dlog}\,\frac{\ell_2}{\ell_0} \wedge \cdots \wedge \mathrm{dlog}\,\frac{\ell_d}{\ell_0}, \tag{2}$$

a wedge of d logarithmic forms with unit residues on the facets $\{\ell_a = 0\}$.

Proposition 3.5 (Triangulation independence, [S]; [1, §2]). *Let P be a convex polytope and $P = \bigcup_\alpha \Delta_\alpha$ a triangulation into simplices meeting on shared faces with matched orientations. Then*

$$\Omega_P \;=\; \sum_\alpha \Omega_{\Delta_\alpha}, \tag{3}$$

and the right-hand side is independent of the triangulation: spurious poles on the internal walls cancel in pairs between adjacent simplices.

The cancellation in Theorem 3.5 is the geometric shadow of a physical fact: internal triangulation walls are not boundaries of P, so they must not appear as poles of the amplitude; unitarity forbids spurious singularities, and the geometry enforces it automatically.

3.3 The residue recursion and the residue tree

Definition 3.6 (Residue tree, [S]). Let $(X, X_{\geq 0}, \Omega_X)$ be a positive geometry of dimension d. Its *residue tree* $\mathsf{R}(X)$ is the rooted tree whose root is X; whose children are the boundary components C_i of $X_{\geq 0}$; and, recursively, whose subtree below each C_i is $\mathsf{R}(C_i)$. Leaves are the 0-dimensional strata (vertices of $X_{\geq 0}$). Equivalently $\mathsf{R}(X)$ is the Hasse diagram of the face poset of $X_{\geq 0}$, oriented from the top face X to the vertices, and decorated at each node by the canonical form of the corresponding stratum.

The residue recursion (1) says exactly that the decoration is compatible with the tree structure: the form at a node determines, by taking residues, the forms at its children. We package this as a functoriality statement.

Proposition 3.7 (Residue tree is functorial under boundary, [S]). *Let* $\mathsf{PosGeom}$ *be the category whose objects are positive geometries and whose morphisms* $X \to X'$ *are the inclusions of boundary strata (each* $C_i \hookrightarrow X$ *a morphism), and let* Tree *be the category of finite rooted trees decorated by rational forms. The assignment* $X \mapsto \mathsf{R}(X)$ *extends to a functor* $\mathsf{R} : \mathsf{PosGeom} \to \mathsf{Tree}$ *such that the residue map* Res_{C_i} *realizes the parent-to-child edge, i.e. the form decorating* C_i *is* Res_{C_i} *of the form decorating* X.

Proof. A boundary inclusion $C_i \hookrightarrow X$ maps to the corresponding edge of $\mathsf{R}(X)$ from the root to the child $\mathsf{R}(C_i)$, which is a rooted-tree morphism (a grafting of subtrees). Composition of boundary inclusions maps to composition of edges, which is associative; identities map to identities. The decoration compatibility is (1). Finiteness of $\mathsf{R}(X)$ follows from finiteness of the face poset of a semialgebraic set with a boundary stratification terminating at points. □

Theorem 3.7 is deliberately modest: it says the residue tree is a well-defined, recursively-decorated combinatorial object, functorial in the obvious way. The content of the regime is what this object computes (Section 4) and what it might secretly equal (Sections 5 and 6).

4 Positive geometries in kinematic space

4.1 Kinematic variables and planar poles

Consider n massless momenta $p_1, \dots, p_n$ with $\sum_i p_i = 0$ and $p_i^2 = 0$. For a fixed cyclic order $(1, 2, \dots, n)$, the planar Mandelstam variables are

$$X_{ij} \;=\; (p_i + p_{i+1} + \cdots + p_{j-1})^2, \qquad 1 \le i < j \le n, \tag{4}$$

with $X_{i,i+1} = 0$ (massless legs) and $X_{1n} = 0$. The nontrivial X_{ij} are indexed by the diagonals (i, j) of a convex n-gon with vertices $1, \dots, n$; there are $\frac{n(n-3)}{2}$ of them. A planar cubic tree with external legs in the cyclic order corresponds to a triangulation of the n-gon; its internal propagators are the diagonals of the triangulation, and it contributes $\prod_{(i,j)\in T} X_{ij}^{-1}$ to the amplitude.

4.2 The ABHY associahedron

Definition 4.1 (ABHY associahedron, [S]; [3]). Fix positive constants $c_{ij} > 0$ for all non-adjacent pairs (i, j) with $1 \le i < j \le n-1$. Inside the linear subspace of kinematic space defined by

$$X_{ij} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j} \;=\; c_{ij}, \tag{5}$$

the *ABHY associahedron* $\mathcal{A}_{n-3}$ is the region cut out by $X_{ij} \ge 0$ for all diagonals (i, j). There are $\frac{(n-2)(n-3)}{2}$ such constraints, cutting the $\frac{n(n-3)}{2}$ planar variables down to an $(n-3)$-dimensional subspace; on it the inequalities carve out a convex polytope of dimension $n-3$, combinatorially the Stasheff associahedron [11]: its vertices are the triangulations of the n-gon and its facets $\{X_{ij} = 0\}$ are the diagonals. (For the pentagon $n = 5$ the non-adjacent pairs with $1 \le i < j \le 4$ are $(1, 3), (1, 4), (2, 4)$: three constraints, leaving the 2-dimensional pentagon $\mathcal{A}_2$.)

Proposition 4.2 (Vertex and facet counts, [S])**.** *The associahedron $\mathcal{A}_{n-3}$ has $C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2}$ vertices (the $(n-2)$-nd Catalan number) and $\frac{n(n-3)}{2}$ facets. For $n=4$: an interval ($C_2 = 2$ vertices, 2 facets). For $n=5$: a pentagon ($C_3 = 5$ vertices, 5 facets). For $n=6$: a three-dimensional associahedron ($C_4 = 14$ vertices, 9 facets).*

Both counts are verified exactly by the companion code (Section A).

Theorem 4.3 (Associahedron canonical form is the bi-adjoint amplitude, [S/H]; [3])**.** *The canonical form of $\mathcal{A}_{n-3}$ is*

$$\Omega_{\mathcal{A}_{n-3}} \;=\; m_n\big(1,\dots,n \,\big|\, 1,\dots,n\big) \;\bigwedge_a dX_a, \tag{6}$$

where the wedge runs over an independent set of $n-3$ diagonal coordinates on the constraint surface (5), and the scalar coefficient

$$m_n \;=\; \sum_{T\in\mathrm{Tri}(n)} \;\prod_{(i,j)\in T} \frac{1}{X_{ij}} \tag{7}$$

is the tree-level color-ordered bi-adjoint scalar (ϕ^3) amplitude, the sum running over triangulations T of the n-gon (equivalently, planar cubic trees). The label S *applies to the geometric identity; the label* H *records that m_n is the physical amplitude only for the bi-adjoint scalar theory (and its double-copy descendants).*

Proof sketch. Each vertex of $\mathcal{A}_{n-3}$ is the transverse intersection of $n-3$ facets $\{X_{ij}=0\}$ whose diagonals form a triangulation T. Near that vertex the polytope looks like a simplicial corner, so by Theorem 3.4 the local contribution to the canonical form is $\pm \bigwedge_{(i,j)\in T} \mathrm{dlog}\, X_{ij}$, which in the coordinates dX_a has coefficient $\prod_{(i,j)\in T} X_{ij}^{-1}$ (up to the constant Jacobian from (5)). Summing the vertex contributions and invoking triangulation independence (Theorem 3.5) gives (7); the spurious poles on internal walls cancel. That the polytope's facet structure matches the diagonals, so that the only poles are the physical $X_{ij} \to 0$, is the ABHY construction [3]. □

Remark 4.4 (Why positivity gives locality and unitarity, [S/H]). Two structural facts fall out. First, the poles of m_n are exactly $X_{ij} \to 0$, the facets of the polytope: these are the physical factorization channels, and there are no others. This is *locality*. Second, the residue on a facet factorizes (Theorem 5.1 below) into a product of lower associahedra, i.e. into a product of lower-point amplitudes. This is *unitarity*. Neither was assumed; both are consequences of the boundary combinatorics of $\mathcal{A}_{n-3}$.

4.3 The amplituhedron and cosmological polytopes

The associahedron is the cleanest positive geometry but not the deepest. Two further constructions extend the reach of the framework.

Definition 4.5 (Amplituhedron, [S/H]; [2])**.** For n particles, k negative-helicity states, and integer m, the (tree) amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is the image of the positive Grassmannian

$\mathrm{Gr}_{\geq 0}(k,n)$ under the linear map induced by a positive matrix $Z \in \mathbb{R}^{n\times(k+m)}$ of external (momentum-twistor) data,

$$\mathrm{Gr}_{\geq 0}(k,n) \longrightarrow \mathrm{Gr}(k,k+m), \qquad C \longmapsto C \cdot Z.$$

It is a positive geometry. For the physical value $m = 4$, its canonical form is the tree integrand of planar $\mathcal{N} = 4$ super-Yang–Mills amplitudes; the loop amplituhedron extends this to all loop orders.

The engine underneath the amplituhedron is the cell decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$ into positroid cells, indexed by plabic graphs / on-shell diagrams and governed by total positivity [9, 10]. The associahedron is the $k = 0$-like "scalar" skeleton of this richer structure. Reviews: Ferro–Lukowski [8] and Herrmann–Trnka [6].

Definition 4.6 (Cosmological polytope, [S/H]; [4])**.** For a graph G with vertices carrying energies x_v and edges carrying energies y_e, the *cosmological polytope* $\mathcal{P}_G$ is the convex hull of the vectors associated to G's subgraphs by a fixed rule. Its canonical form computes the wavefunction-of-the-universe coefficient for a conformally-coupled scalar with polynomial interactions in a class of FRW cosmologies. This is the gravitational/cosmological extension of the amplitude story, valid for the stated toy models.

Theorem 4.6 is the reason the amplitude regime is not only a gauge-theory story. The cosmological polytope reconstructs a cosmological observable—a boundary wavefunction coefficient, directly relevant to inflationary correlators—as the canonical form of a polytope, with no de Sitter Lagrangian or in–in path integral in the definition. The caveat, which we do not hide, is that the models are conformally-coupled scalars, not Einstein gravity; the extension to realistic FRW backgrounds and to loop level is active and incomplete.

5 Main results

We now state the paper's results. Theorem 5.1 and Theorem 5.2 are standard mathematics [S]. Theorem 5.3 is standard [S]. The bridge statements Theorem 5.6 and Theorem 5.7 are, respectively, the field's central conjecture [H/P] and a conditional theorem [S/H] that makes precise what a proof would have to supply.

5.1 Factorization from residues

Theorem 5.1 (Residue factorization on associahedron facets, [S])**.** *Let (i,j) be a diagonal of the n-gon, splitting it into a sub-polygon L on vertices $\{i, i+1, \ldots, j\}$ and a sub-polygon R on vertices $\{j, j+1, \ldots, n, 1, \ldots, i\}$, each with the edge (i,j) adjoined. Then the facet $\{X_{ij} = 0\}$ of $\mathcal{A}_{n-3}$ is the product $\mathcal{A}_L \times \mathcal{A}_R$ of the two sub-associahedra, and*

$$\mathrm{Res}_{X_{ij}=0}\, m_n \;=\; m_L \cdot m_R, \tag{8}$$

where m_L, m_R are the bi-adjoint amplitudes of the sub-polygons. Equivalently, at the level of forms,

$$\mathrm{Res}_{X_{ij}=0}\, \Omega_{\mathcal{A}_{n-3}} \;=\; \Omega_{\mathcal{A}_L} \wedge \Omega_{\mathcal{A}_R}.$$

Proof. A triangulation T of the n-gon contains the diagonal (i, j) if and only if it restricts to a triangulation T_L of L and T_R of R; conversely any pair (T_L, T_R) glues, with (i, j), to such a T. Hence the triangulations through (i, j) are in bijection with $\mathrm{Tri}(L) \times \mathrm{Tri}(R)$. In (7) the terms with a pole at $X_{ij} = 0$ are exactly those $T \ni (i, j)$, each of the form $X_{ij}^{-1} \prod_{(a,b)\in T_L} X_{ab}^{-1} \prod_{(c,d)\in T_R} X_{cd}^{-1}$. Taking the residue at $X_{ij} = 0$ removes the X_{ij}^{-1} and sums over (T_L, T_R), giving $m_L \cdot m_R$. Geometrically, the facet $\{X_{ij} = 0\}$ of the associahedron is the product polytope $\mathcal{A}_L \times \mathcal{A}_R$, and the residue of a canonical form on a product facet is the product (wedge) of the factor canonical forms. $\square$

Theorem 5.1 is the precise sense in which unitarity is a theorem about a polytope. We verify (8) exactly, as an identity of sets of triangulations and numerically, for the pentagon in Section 6 and in the companion code.

Proposition 5.2 (Leaves are cubic trees, [S]). *The leaves of the residue tree* $\mathsf{R}(\mathcal{A}_{n-3})$ *are the vertices of the associahedron, hence the triangulations of the n-gon, hence the planar cubic trees with external legs* $1, \ldots, n$ *in cyclic order. Each root-to-leaf path in* $\mathsf{R}(\mathcal{A}_{n-3})$ *is a maximal chain of compatible (non-crossing) diagonals, i.e. a nested sequence of factorization channels ending on a single Feynman-like tree.*

Proof. Immediate from Theorem 3.6 and Theorem 4.1: faces of $\mathcal{A}_{n-3}$ of codimension c are the sets of c pairwise non-crossing diagonals (partial triangulations), with vertices the maximal such sets (full triangulations). A root-to-leaf path adds one compatible diagonal at a time, terminating at a triangulation. $\square$

Proposition 5.3 (Residue-decorated tree, [S]). *The functor* R *of Theorem 3.7, applied to* $\mathcal{A}_{n-3}$*, decorates each node—a partial triangulation S (a face)—by the canonical form of that face, which by iterated application of Theorem 5.1 is a product* $\prod_p m_{P_p}$ *of bi-adjoint amplitudes of the polygons* P_p *into which the diagonals of S dissect the n-gon. The edge from a node S to a child* $S \cup \{(i, j)\}$ *is realized by* $\mathrm{Res}_{X_{ij}=0}$.

Proof. By induction on codimension using Theorem 5.1. A face S of codimension c is a set of c non-crossing diagonals dissecting the n-gon into $c + 1$ sub-polygons $P_0, \ldots, P_c$; the corresponding sub-associahedron is $\prod_p \mathcal{A}_{P_p}$ with canonical form $\bigwedge_p \Omega_{\mathcal{A}_{P_p}}$, i.e. coefficient $\prod_p m_{P_p}$. Adding a compatible diagonal (i, j) subdivides one P_p and, by Theorem 5.1, the residue at $X_{ij} = 0$ splits its factor m_{P_p} into $m_L \cdot m_R$. $\square$

5.2 The coaction side and the bridge

We now recall the coaction tree and state the bridge conjecture.

Definition 5.4 (Motivic coaction and coaction tree, [H]). Let $\mathcal{H}$ be the Hopf algebra of motivic multiple polylogarithms (or motivic multiple zeta values in the constant case), graded by weight, and let $\mathcal{A} = \mathcal{H}/(\text{weight} > 0)^2$ be its Lie-coalgebra "de Rham" quotient. The motivic coaction is a coassociative map

$$\Delta : \mathcal{H} \longrightarrow \mathcal{A} \otimes \mathcal{H}, \tag{9}$$

lowering the weight of the left factor. Iterating the reduced coaction $\Delta' = \Delta - 1 \otimes \mathrm{id}$ produces, for a weight-w period F, a rooted tree $\mathsf{C}(F)$: the root is F; the children are the terms of $\Delta' F$; recursing on the right (lower-weight) factors terminates at weight-one primitives (logarithms). The cobracket $\delta : \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}$ on the Lie coalgebra $\mathcal{L}$ records the antisymmetric part.

Example 5.5 (Low-weight coaction trees, [H])**.** The classical dilogarithm $\mathrm{Li}_2(x)$ (weight 2) has reduced coaction $\Delta' \mathrm{Li}_2^{\mathfrak{m}}(x) = -\log^{\mathfrak{m}}(1-x) \otimes \log(x)$ modulo lower structure; its cobracket is $\delta\, \mathrm{Li}_2(x) = -\log(1-x) \wedge \log(x)$. So $\mathsf{C}(\mathrm{Li}_2(x))$ is a root with a single weight-$(1,1)$ child, whose two factors are the weight-one leaves $\log(1-x), \log(x)$. The motivic $\zeta^{\mathfrak{m}}(2)$ is coaction-primitive: $\Delta' \zeta^{\mathfrak{m}}(2) = 0$ (it behaves like a power of $2\pi i$), so $\mathsf{C}(\zeta(2))$ is a single node. The motivic $\zeta^{\mathfrak{m}}(3)$ is primitive of weight three: $\Delta' \zeta^{\mathfrak{m}}(3) = 0$ in the reduced coaction and $\mathsf{C}(\zeta(3))$ is again a single (weight-three) node.

Conjecture 5.6 (Residue tree = coaction tree, [H/P]; open-problem queue item 1)**.** *Let X be a positive geometry whose canonical form, integrated against a suitable (twisted) measure, yields a period F_X in the mixed-Tate class. Then the residue tree $\mathsf{R}(X)$ and the coaction tree $\mathsf{C}(F_X)$ are isomorphic as rooted trees, by an isomorphism matching each boundary stratum of X with a coaction term of F_X of the corresponding weight, compatibly with the residue and coaction edges:*

$$\mathsf{R}(X) \xrightarrow{\cong} \mathsf{C}(F_X), \qquad \begin{array}{ccc} X & \xrightarrow{\mathrm{Res}_{C_i}} & C_i \\ {\scriptstyle\cong}\big\downarrow & & \big\downarrow{\scriptstyle\cong} \\ F_X & \xrightarrow{\Delta'\text{-}term} & F_{C_i} \end{array} \tag{10}$$

The status is H *for the verified low-weight, low-multiplicity, mixed-Tate cases and* P *for the general statement.*

The evidence for Theorem 5.6 is structural and case-based rather than a general theorem, and we are explicit about that. Structurally: cluster coordinates on $\mathrm{Gr}(k,n)$ simultaneously coordinatize the boundary strata of the positive geometry and the arguments (letters) of the polylogarithms appearing in the amplitude, so the two decompositions are indexed by the same combinatorial data [13]. Case-based: the coaction of one- and two-loop amplitudes has been matched to the symbol/cluster structure in explicit examples, and the empirical "coaction principle" for Feynman periods [15, 16] shows the amplitude class is closed under Δ, a necessary condition for the trees to align. The cleanest positive statement we can make is conditional and rests on the twisted-cohomology coaction of Abreu–Britto–Duhr–Gardi–Matthew [18].

Theorem 5.7 (Conditional bridge via twisted cohomology, [S/H])**.** *Suppose $F_X = \int_\gamma u\, \varphi$ is a period in which the integrand φ is the canonical form of a positive geometry, the contour γ is (the positive part of) a positive geometry, and $u = \prod_a \ell_a^{\epsilon n_a}$ is a twist regulating the boundary singularities by exponents proportional to a small parameter ϵ. Let Δ_ϵ be the twisted-cohomology coaction of [18], whose left ("master contour") factor ranges over a basis of twisted cycles dual to the boundary strata of γ. Then:*

(i) the left factors of $\Delta_\epsilon F_X$ are indexed by the codimension-one boundary strata of the contour γ, i.e. by the depth-one children of the residue tree $\mathsf{R}(\gamma)$;

(ii) iterating Δ_ϵ produces a tree whose nodes at depth c are indexed by the codimension-c strata, so that the twisted coaction tree $\mathsf{C}_\epsilon(F_X)$ is isomorphic to the residue tree $\mathsf{R}(\gamma)$;

(iii) as $\epsilon \to 0$ the twisted coaction Δ_ϵ reduces to the motivic coaction Δ on multiple polylogarithms, matching (9) order by order in ϵ.

Consequently, in the regularized (twisted) category the residue tree is *the coaction tree ($\mathsf{R}(\gamma) \cong \mathsf{C}_\epsilon(F_X)$, an S statement); the H content is that the $\epsilon \to 0$ limit preserves the isomorphism onto the motivic coaction tree $\mathsf{C}(F_X)$, which is what Theorem 5.6 asserts and which is proven only in special cases.*

Discussion. Parts (i)–(iii) are the content of [18]: they construct Δ_ϵ using intersection numbers of twisted cocycles and cycles, with the "contour" half of the coaction valued in twisted homology, whose natural basis is dual to the boundary stratification of γ; the master decomposition therefore mirrors the residue tree. The compatibility with the polylogarithm coaction as $\epsilon \to 0$ is their consistency theorem. What is *not* supplied—and is the gap between Theorem 5.7 and Theorem 5.6—is a proof that the tree isomorphism survives the limit for an arbitrary positive geometry and yields the motivic coaction tree without collapse or refinement of nodes. In the mixed-Tate low-weight cases the limit is controlled and the trees match; in general (and certainly in the non-Tate case, Section 8) it is open. Hence the composite status S/H, degrading to H/P once the unregularized general claim of Theorem 5.6 is asserted. □

Theorem 5.7 is the honest form of the bridge: there is a category (twisted cohomology, $\epsilon \neq 0$) in which the two trees provably coincide, and the open problem is the fate of that coincidence in the $\epsilon \to 0$ motivic limit.

6 Worked example: the pentagon

We now carry out, in full, the five-point case: the geometry of the pentagon associahedron, its canonical form, an explicit residue computation exhibiting factorization, its residue tree, and the residue-tree/coaction-tree comparison.

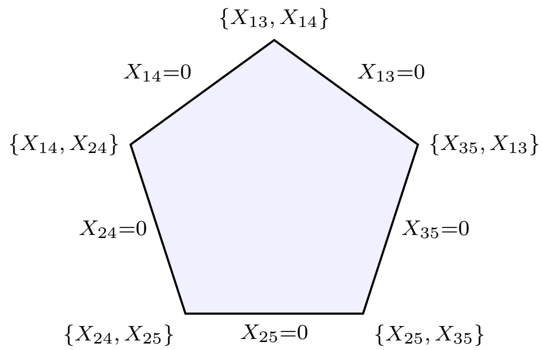
6.1 The pentagon and its canonical form

For $n = 5$ the diagonals of the pentagon 12345 are

$$X_{13},\ X_{14},\ X_{24},\ X_{25},\ X_{35}, \tag{11}$$

five in number, matching $\frac{5\cdot 2}{2} = 5$. The associahedron $\mathcal{A}_2$ is a pentagon: a two-dimensional polytope with five vertices and five edges (facets). The five vertices are the five triangulations of the pentagon; each uses two non-crossing diagonals. Two diagonals $(i,j),(k,l)$ cross iff their endpoints interleave around the pentagon. The non-crossing (compatible) pairs are exactly the cyclically consecutive ones:

$$\{X_{13}, X_{14}\},\ \{X_{14}, X_{24}\},\ \{X_{24}, X_{25}\},\ \{X_{25}, X_{35}\},\ \{X_{35}, X_{13}\}. \tag{12}$$



The pentagon associahedron $\mathcal{A}_2$. Vertices are triangulations (pairs of compatible diagonals); an edge is a facet $\{X_{ij} = 0\}$, labelled by the single diagonal whose vanishing it enforces; the two endpoints of that edge are the two triangulations containing X_{ij}.

By Theorem 4.3, summing the five vertex contributions gives the canonical-form coefficient

$$m_5 \;=\; \frac{1}{X_{13}X_{14}} + \frac{1}{X_{14}X_{24}} + \frac{1}{X_{24}X_{25}} + \frac{1}{X_{25}X_{35}} + \frac{1}{X_{35}X_{13}}. \tag{13}$$

This is the five-point bi-adjoint scalar amplitude. It has exactly five simple poles, one per diagonal, and no others—in particular no pole at any *crossing* pair such as $\{X_{13}, X_{24}\}$, which would be a spurious (non-planar) singularity. The absence of spurious poles is triangulation independence (Theorem 3.5) made concrete.

6.2 An explicit residue: factorization on $X_{13} = 0$

Take the residue of m_5 at the facet $X_{13} = 0$. The terms of (13) with a pole there are the first and the last:

$$\mathrm{Res}_{X_{13}=0}\; m_5 \;=\; \mathrm{Res}_{X_{13}=0}\left(\frac{1}{X_{13}X_{14}} + \frac{1}{X_{35}X_{13}}\right) \;=\; \frac{1}{X_{14}} + \frac{1}{X_{35}}. \tag{14}$$

Now interpret the right-hand side. The diagonal $(1,3)$ splits the pentagon 12345 into a triangle $L = (1,2,3)$ and a quadrilateral $R = (1,3,4,5)$. The triangle is $\mathcal{A}_0$, a point, with $m_L = 1$ (a three-point amplitude, no propagator). The quadrilateral is $\mathcal{A}_1$, an interval, with two diagonals $(1,4)$ and $(3,5)$ and amplitude

$$m_R \;=\; \frac{1}{X_{14}} + \frac{1}{X_{35}}. \tag{15}$$

Thus (14) reads

$$\mathrm{Res}_{X_{13}=0}\; m_5 \;=\; m_L \cdot m_R \;=\; 1 \cdot \left(\frac{1}{X_{14}} + \frac{1}{X_{35}}\right), \tag{16}$$

an instance of Theorem 5.1: the residue on the X_{13} channel factorizes into the three-point amplitude (trivial) times the four-point sub-amplitude. Physically, the internal line with momentum $p_1 + p_2$ goes on shell and the process splits into a $1, 2 \to I$ vertex and an $I, 3, 4, 5$ sub-amplitude. The companion code verifies (14), both as an identity of triangulation sets and numerically at sample kinematics, for all five facets.

6.3 The residue tree of the pentagon

Assembling the residues gives the residue tree $\mathsf{R}(\mathcal{A}_2)$:

$$\begin{tikzcd}
 & & \mathcal{A}_2\ (m_5) \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
\frac{1}{X_{14}}+\frac{1}{X_{35}} & \frac{1}{X_{13}}+\frac{1}{X_{24}} & \frac{1}{X_{14}}+\frac{1}{X_{25}} & \frac{1}{X_{24}}+\frac{1}{X_{35}} & \frac{1}{X_{25}}+\frac{1}{X_{13}}
\end{tikzcd}$$

Root: the pentagon, decorated by m_5. Depth one, left to right: the five facets $X_{13} = 0$, $X_{14} = 0$, $X_{24} = 0$, $X_{25} = 0$, $X_{35} = 0$, each an interval $\mathcal{A}_1$ decorated by the corresponding four-point sub-amplitude (the residue of m_5 there). Depth two (not drawn): each interval has two vertices, 0-dimensional strata decorated by 1, the leaves; these are the five triangulations of the pentagon. Counting faces gives the f-vector $(5, 5, 1)$ of the pentagon: five vertices, five edges, one 2-face.

The five facet residues, read off as in (14) for each diagonal, are the five displayed nodes: $\mathrm{Res}_{X_{13}=0}\, m_5 = \frac{1}{X_{14}} + \frac{1}{X_{35}}$, $\mathrm{Res}_{X_{14}=0}\, m_5 = \frac{1}{X_{13}} + \frac{1}{X_{24}}$, $\mathrm{Res}_{X_{24}=0}\, m_5 = \frac{1}{X_{14}} + \frac{1}{X_{25}}$, $\mathrm{Res}_{X_{25}=0}\, m_5 = \frac{1}{X_{24}} + \frac{1}{X_{35}}$, and $\mathrm{Res}_{X_{35}=0}\, m_5 = \frac{1}{X_{25}} + \frac{1}{X_{13}}$. Each is a four-point amplitude on the appropriate sub-quadrilateral. This is the complete factorization data of m_5, encoded geometrically.

6.4 The coaction-tree comparison

The residue tree above is a tree of *rational* functions: m_5 has weight zero, so its motivic coaction is trivial and there is, at this level, nothing on the coaction side to compare. The nontrivial comparison requires promoting the canonical form to a transcendental period, which is exactly what the stringy / twisted completion does. We describe the comparison honestly, tagging each step.

Step 1: the stringy pentagon integral `[S]`**.** Replace the rational canonical form by the Koba–Nielsen (string) integral over the same positive geometry. Concretely, integrate the canonical form of the moduli space $\mathcal{M}_{0,5}$ (the worldsheet positive geometry whose boundary combinatorics is the same associahedron) against the Koba–Nielsen twist $u = \prod_{i<j} |z_{ij}|^{\alpha' s_{ij}}$. The result is the five-point open-string (Veneziano-type) partial amplitude $A_5(\alpha')$, a function of the Mandelstam invariants whose low-energy expansion is a power series in α' with multiple-zeta-value coefficients:

$$A_5(\alpha') \;=\; m_5 \;+\; \alpha'^2\, \zeta(2)\, (\cdots) \;+\; \alpha'^3\, \zeta(3)\, (\cdots) \;+\; \cdots\,, \tag{17}$$

the leading term being the field-theory amplitude (13). This is standard.

Step 2: the coaction of the string integral `[S/H]`**.** By Theorem 5.7 (the construction of [18], and the genus-zero configuration-space setting), the twisted coaction Δ_ϵ (here $\epsilon \sim \alpha'$) acts on A_5 with a contour factor ranging over the boundary strata of the moduli-space contour—i.e. over the facets of the associahedron, the same five diagonals X_{ij}. The depth-one children of the twisted coaction tree $\mathsf{C}_\epsilon(A_5)$ are therefore indexed by the five facets, in bijection with the depth-one children of the residue tree $\mathsf{R}(\mathcal{A}_2)$. At this regularized level the two trees agree (S), and the identification of the fifth facet, its residue, and its twisted-cycle dual is explicit.

Step 3: the motivic limit [H]**.** Expanding in α' and taking $\epsilon \to 0$, the twisted coaction reduces to the motivic coaction on the MZV coefficients of (17). At weight two the coefficient is $\zeta(2)$, coaction-primitive (Theorem 5.5): a single node, matching a single facet-residue at leading transcendental order. At weight three the coefficient is $\zeta(3)$, again primitive: a single node. At these low weights the coaction tree of A_5 is compatible, node for node, with the residue tree of $\mathcal{A}_2$: the boundary strata carry the same recursive structure as the successive coaction terms. This is the concrete, low-weight instance of Theorem 5.6 for $n = 5$, and it is H: verified in the mixed-Tate low-weight sector, not proven to persist to all weights.

Step 4: what would break [P]**.** Beyond weight three, MZVs acquire nontrivial coaction structure (e.g. $\zeta(3) \otimes \zeta(2)$-type terms in weight five, depth-two coaction trees), and the question becomes whether the deepening of the coaction tree continues to track the deepening of the residue tree under further residues. For the pentagon the residue tree has depth two (it is a polygon), while the coaction tree grows without bound in weight; the correct statement is not that the finite residue tree equals an infinite coaction tree, but that the residue tree controls the *first* layers and that a graded/filtered refinement is needed to state the match precisely. Making this precise—and, above all, extending it past the mixed-Tate regime—is exactly open-problem item 1, and it is here P.

Remark 6.1 (What the pentagon does and does not establish)*.* The pentagon establishes, rigorously ([S]), the geometry of $\mathcal{A}_2$, its canonical form m_5, the residue factorization (14), and the residue tree. It establishes, in a regularized category ([S/H]), that the boundary strata index the depth-one coaction terms of the string completion. It illustrates, at low weight ([H]), the residue-tree/ coaction-tree match. It does *not* establish ([P]) the match to all weights or outside the mixed-Tate regime. This layered honesty is the point of the S/H/P tags.

7 Relation to neighbouring regimes

7.1 Double copy: gravity through the inverse associahedron amplitude

The bi-adjoint amplitude m_n—the associahedron canonical form—is not only a scalar toy. It is the kernel through which gauge-theory amplitudes are squared into gravity. The field-theory Kawai–Lewellen–Tye (KLT) relations [21, 20] express an n-point (super)gravity tree amplitude as a bilinear in color-ordered gauge-theory amplitudes,

$$M_n^{\text{grav}} \;=\; \sum_{\alpha,\beta} A_n(\alpha)\, S[\alpha|\beta]\, \widetilde{A}_n(\beta), \tag{18}$$

where the momentum kernel $S[\alpha|\beta]$ is, up to normalization, the inverse of the matrix of bi-adjoint amplitudes,

$$S = m^{-1}, \qquad m_{\alpha\beta} = m_n(\alpha|\beta), \tag{19}$$

the double-color-ordered bi-adjoint amplitudes [22]. Since each $m_n(\alpha|\beta)$ is the canonical form of an associahedron (in the pair of orderings α, β), the entire double copy is phrased through associahedron canonical forms: gravity is gauge $\otimes\, m^{-1} \otimes$ gauge. This is the sharp sense in which the gravitational S-matrix is reconstructed from gauge-theory representation data, with the positive geometry of the scalar theory supplying the "glue."

Proposition 7.1 (Status of the double-copy bridge, [S/H])**.** *The KLT/BCJ double copy* (18)–(19) *is standard mathematics at tree level given color–kinematics-dual numerators* [S]*, and a heuristic (verified, not proven in general) construction at loop level* [H]*. Composing with the positive-geometry entry* S/H *leaves the status at* S/H *for tree-level statements and* H *once loop-level double copy is invoked, by the worst-component rule of Theorem 2.2.*

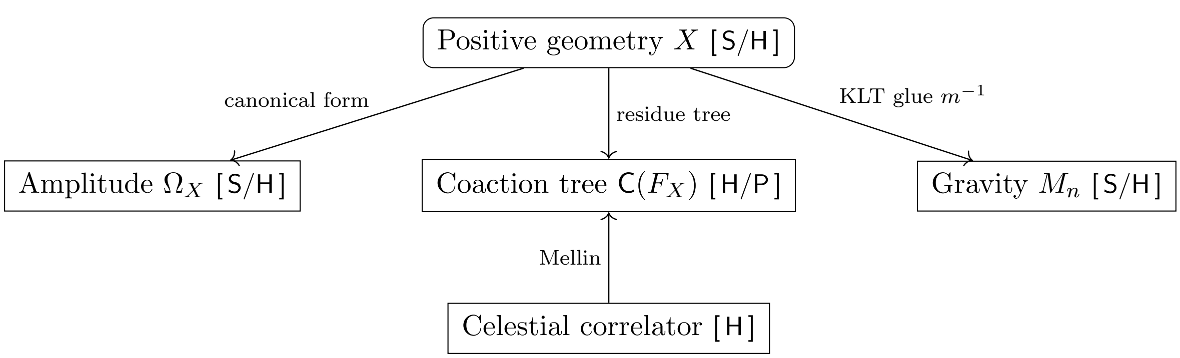
7.2 Motivic coactions

The relation to the motivic-coaction regime is the content of Sections 5 and 6: the residue tree of the positive geometry and the coaction tree of its period. The composition of the positive-geometry entry S/H with the motivic-coaction entry H gives H for the bridge in the mixed-Tate regime, degrading to P in general. The coaction principle for Feynman periods [15, 16] is the empirical backbone: the amplitude class is (conjecturally, with strong evidence) closed under Δ, which is a prerequisite for any residue-tree/coaction-tree isomorphism.

7.3 Celestial holography

The Mellin transform in each particle's energy converts momentum eigenstates into conformal primaries on the celestial sphere [23, 24], reorganizing the flat-space amplitude as a celestial correlator. This is a change of representation basis for the one-particle states (an $\mathrm{SL}(2,\mathbb{C})$ representation-theoretic move), applied to the same S-matrix whose integrand is the canonical form. Composing the celestial entry H with the positive-geometry entry keeps the composite at H: the celestial repackaging does not upgrade the positive-geometry construction to a full quantum-gravity statement, and the project keeps celestial holography at separate status from AdS/CFT-style holography by policy.

7.4 The composed picture



The amplitude regime and its neighbours, with composed statuses. Every arrow degrades or preserves status by the worst-component rule; none upgrades it.

8 Limitations and open problems

We collect the honest obstructions. None is hidden in the abstract or the introduction.

8.1 Elliptic and non-Tate amplitudes [H/P]

The motivic coaction of Theorem 5.4 lives in the tame, mixed-Tate world: its periods are (multiple) polylogarithms and MZVs, and the coaction is the Goncharov–Brown coaction on that Hopf algebra. Many physical amplitudes leave this world. Already the two-loop sunrise integral and the ten-point two-loop $\mathcal{N} = 4$ amplitude involve elliptic polylogarithms and periods of elliptic curves (and, beyond, of higher-genus curves and Calabi–Yau varieties). For these the mixed-Tate coalgebra is the wrong target; one needs an enlarged (elliptic/modular) coalgebra, and the coaction tree of Theorem 5.6 must be reformulated accordingly. This is open-problem item 3 of the project queue. Until the non-Tate coaction is under control, the residue-tree/coaction-tree bridge is at best H in the mixed-Tate sector and P outside it. The positive geometry, by contrast, is often still well defined (the elliptic obstruction lives on the period/coaction side, not necessarily on the boundary-stratification side), which sharpens the open problem: the residue tree may exist where the coaction tree does not yet.

8.2 Gravity versus gauge positivity [P]

The amplituhedron and associahedron encode gauge-theory (and bi-adjoint scalar) data through *positivity*: the positive Grassmannian, positive kinematic constants, convex polytopes. There is no established positive geometry whose canonical form is a general gravitational integrand. Gravity enters this regime only indirectly: through the double copy (Equation (18)), which reconstructs gravity from gauge data plus the scalar glue m^{-1}, and through cosmological polytopes (Theorem 4.6), which cover conformally-coupled toy models rather than Einstein gravity. Whether "gravity has its own positive geometry"—a convex/positive object whose canonical form is the graviton amplitude directly, not via squaring—is open and, we judge, P. The tension is structural: gravitational amplitudes have double poles and worse in convenient variables, and their numerators are not manifestly positive, so the naive positivity that powers the gauge-theory constructions does not transfer.

8.3 Generality of the S-matrix [P]

Even within gauge theory, the sharpest results (amplituhedron = planar $\mathcal{N} = 4$ integrand) are for a maximally supersymmetric, planar theory. Non-planar corrections, reduced supersymmetry, and massive external states are the subject of active construction (momentum amplituhedron, associahedra for ϕ^p, massive deformations) but are not uniformly under control. The statement "amplitude = canonical form" is a theorem in a growing but bounded list of theories, not a universal law. We keep the positive-geometry entry at S/H precisely to record this: S where the construction is proven, H as a claim about physics in general.

8.4 The bridge itself [H/P]

Finally, Theorem 5.6 is a conjecture. Theorem 5.7 shows a regularized category where the two trees coincide, and the pentagon (Section 6) shows the low-weight match, but a general proof—especially one surviving the $\epsilon \to 0$ motivic limit and the non-Tate extension—does not exist. This is the central open problem of the regime, and stating it cleanly, with the twisted-cohomology construction as the most concrete foothold, is the main conceptual contribution of this paper.

9 Conclusion

We have presented perturbative amplitudes as canonical forms of positive geometries, with the recursive residue/boundary stratification—the residue tree—as the core invariant. The framework reconstructs an amplitude from boundary and positivity data alone: no Lagrangian, no Feynman sum, no path integral enters the definition of the canonical form, and locality and unitarity emerge as theorems about a polytope (Theorem 5.1). We worked the pentagon associahedron in full, computed an explicit residue exhibiting factorization, drew its residue tree, and compared that tree to the motivic coaction tree of the string/twisted completion, using the twisted-cohomology coaction as the concrete bridge (Theorem 5.7). We tracked the epistemic status of every claim under the worst-component-wins calculus: the geometry and residue recursion are standard S; the identification with a physical S-matrix is theory-specific S/H; and the residue-tree/coaction-tree bridge is heuristic in the mixed-Tate low-weight regime and speculative in general, H/P.

The reconstruction thesis is unusually well served here: in the theories where it holds, the amplitude simply *is* the canonical form, and gravity enters through the double copy as gauge data glued by the inverse associahedron amplitude. The honest frontier is threefold: the non-Tate (elliptic and beyond) extension of the coaction side, the absence of a direct gravitational positive geometry, and a general proof of the residue-tree/coaction-tree isomorphism. The companion Haskell development makes the combinatorial and residue content of the pentagon (and general associahedron) exactly checkable, closing the loop between the geometry and a runnable computation.

A Companion Haskell development

The code in `src/positive-geometries/` accompanies this paper. It is written against the GHC `base` library only (no external dependencies) and compiles with `ghc`. The modules are:

- `Core.hs` — the associahedron combinatorics: convex polygons, diagonals (i, j), the non-crossing (compatibility) test, enumeration of triangulations of an n-gon, and the bi-adjoint amplitude m_n of (7) represented symbolically as a list of triangulations (each a set of diagonals, i.e. a product of propagators). Residue extraction $\mathrm{Res}_{X_{ij}=0}$ is implemented as: keep the triangulations containing the diagonal (i, j), delete (i, j) from each. A weight-graded toy coalgebra element with a one-step coproduct is included to illustrate the coaction tree of Theorem 5.4 and the status calculus $\star$ of Theorem 2.2.

- `Properties.hs` — exact, exhaustive checks over the finite combinatorics: (i) the number of triangulations of the n-gon equals the Catalan number C_{n-2} (Theorem 4.2); (ii) the number of diagonals equals $\frac{n(n-3)}{2}$; (iii) the residue factorization $\mathrm{Res}_{X_{ij}=0}\, m_n = m_L \cdot m_R$ of Theorem 5.1, checked for the pentagon on all five facets both as an identity of triangulation sets and numerically at sample kinematics; (iv) the alternating f-vector sum (Euler characteristic) of the pentagon; (v) the status-calculus laws (unit $\mathsf{S} \star \sigma = \sigma$ and monotonicity of Theorem 2.2).

- `Main.hs` — a `main` that prints the pentagon's diagonals, triangulations, canonical-form terms m_5 of (13), the explicit residue (14) and its factorized form, and then runs the verification suite from `Properties.hs`, exiting non-zero on any failure.

The checkable property singled out for the paper is Theorem 5.1 on the pentagon: $\mathrm{Res}_{X_{13}=0}\, m_5 = \frac{1}{X_{14}} + \frac{1}{X_{35}}$, verified as (14).

References

[1] N. Arkani-Hamed, Y. Bai, and T. Lam, *Positive Geometries and Canonical Forms*, JHEP **11** (2017) 039; arXiv:1703.04541 [hep-th].

[2] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, JHEP **10** (2014) 030; arXiv:1312.2007 [hep-th].

[3] N. Arkani-Hamed, Y. Bai, S. He, and G. Yan, *Scattering Forms and the Positive Geometry of Kinematics, Color and the Worldsheet*, JHEP **05** (2018) 096; arXiv:1711.09102 [hep-th].

[4] N. Arkani-Hamed, P. Benincasa, and A. Postnikov, *Cosmological Polytopes and the Wavefunction of the Universe*, arXiv:1709.02813 [hep-th].

[5] T. Lam, *An Invitation to Positive Geometries*, in *Open Problems in Algebraic Combinatorics*, Proc. Sympos. Pure Math. (2024), 159–179; arXiv:2208.05407 [math.CO].

[6] E. Herrmann and J. Trnka, *The SAGEX Review on Scattering Amplitudes, Chapter 7: Positive Geometry of Scattering Amplitudes*, J. Phys. A **55** (2022) 443008; arXiv:2203.13018 [hep-th].

[7] F. Brown and C. Dupont, *Positive Geometries and Canonical Forms via Mixed Hodge Theory*, Commun. Math. Phys. **406** (2025); DOI 10.1007/s00220-025-05399-y; arXiv:2501.03202 [math.AG].

[8] Ø. Ferro and T. Łukowski, *Amplituhedra, and Beyond*, J. Phys. A **53** (2020) 033001; arXiv:2007.04342 [hep-th].

[9] A. Postnikov, *Total Positivity, Grassmannians, and Networks*, arXiv:math/0609764 [math.CO].

[10] N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes*, Cambridge University Press (2016); arXiv:1212.5605 [hep-th].

[11] J. Stasheff, *Homotopy Associativity of H-Spaces I*, Trans. Amer. Math. Soc. **108** (1963) 275–292.

[12] A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Classical Polylogarithms for Amplitudes and Wilson Loops*, Phys. Rev. Lett. **105** (2010) 151605; arXiv:1006.5703 [hep-th].

[13] J. Golden, A. B. Goncharov, M. Spradlin, C. Vergu, and A. Volovich, *Motivic Amplitudes and Cluster Coordinates*, JHEP **01** (2014) 091; arXiv:1305.1617 [hep-th].

[14] F. Brown, *Mixed Tate Motives over $\mathbb{Z}$*, Ann. of Math. **175** (2012) 949–976; arXiv:1102.1312 [math.AG].

[15] F. Brown, *Feynman Amplitudes, Coaction Principle, and Cosmic Galois Group*, Commun. Number Theory Phys. **11** (2017) 453–556; arXiv:1512.06409 [math-ph].

[16] E. Panzer and O. Schnetz, *The Galois Coaction on ϕ^4 Periods*, Commun. Number Theory Phys. **11** (2017) 657–705; arXiv:1603.04289 [hep-th].

[17] C. Duhr, *Hopf Algebras, Coproducts and Symbols: an Application to Higgs Boson Amplitudes*, JHEP **08** (2012) 043; arXiv:1203.0454 [hep-th].

[18] S. Abreu, R. Britto, C. Duhr, E. Gardi, and J. Matthew, *From Positive Geometries to a Coaction on Hypergeometric Functions*, JHEP **02** (2020) 122; arXiv:1910.08358 [hep-th].

[19] Z. Bern, J. J. M. Carrasco, and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, Phys. Rev. D **78** (2008) 085011; arXiv:0805.3993 [hep-ph].

[20] Z. Bern, J. J. M. Carrasco, and H. Johansson, *Perturbative Quantum Gravity as a Double Copy of Gauge Theory*, Phys. Rev. Lett. **105** (2010) 061602; arXiv:1004.0476 [hep-th].

[21] H. Kawai, D. C. Lewellen, and S.-H. H. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, Nucl. Phys. B **269** (1986) 1–23.

[22] F. Cachazo, S. He, and E. Y. Yuan, *Scattering of Massless Particles: Scalars, Gluons and Gravitons*, JHEP **07** (2014) 033; arXiv:1309.0885 [hep-th].

[23] S. Pasterski, S.-H. Shao, and A. Strominger, *Flat Space Amplitudes and Conformal Symmetry of the Celestial Sphere*, Phys. Rev. D **96** (2017) 065026; arXiv:1701.00049 [hep-th].

[24] A. Strominger, *Lectures on the Infrared Structure of Gravity and Gauge Theory*, Princeton University Press (2018); arXiv:1703.05448 [hep-th].