# Geometry as a Theorem About an Algebra: Spectral Triples, the Connes Distance Formula, and the Reconstruction of Metric Spacetime from Spectral Data

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**Abstract**

We treat one regime of a modular reconstruction program in which spacetime, matter, and causality are recovered as invariants of representation data rather than obtained by quantizing fields on a fixed background. The regime here is metric spin geometry, and the carrier of its content is Connes's spectral triple $(\mathcal{A}, \mathcal{H}, D)$: a $*$-algebra $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$ together with a self-adjoint operator $D$ with compact resolvent and bounded commutators $[D, a]$. The reconstruction thesis becomes a precise statement. First, Gelfand–Naimark duality recovers the *points* of a space from a commutative $C^*$-algebra. Second, the Connes distance formula $d(p, q) = \sup\{|p(a) - q(a)| : \|[D, a]\| \leq 1\}$ recovers the *geodesic metric* from the pair $(\mathcal{A}, D)$. Third, Connes's reconstruction theorem (2013) recovers the entire Riemannian spin$^c$ manifold, as a theorem, from a commutative real spectral triple satisfying an explicit axiom list, with no manifold assumed anywhere in the input. We give self-contained statements of the spectral-triple axioms, the distance formula, the reconstruction theorem, and the Chamseddine–Connes spectral action, and we carry out two computations in full: the canonical Dirac triple of a Riemannian spin manifold, where the distance formula returns the geodesic distance; and the two-point finite spectral triple, where an off-diagonal Dirac matrix of modulus $m$ produces the metric $d(1, 2) = 1/m$. The second computation is verified by an accompanying Haskell program. We label every claim with the project's S/H/P calculus, which is monotone under composition: the kinematic reconstruction theorems are rigorous (S), while the dynamical and phenomenological uses of the framework are model dependent (H), so that no chain routed through this regime is rated above H, and the specific chain that would give a Lorentzian reconstruction of spacetime drops to

H/P because the required Lorentzian reconstruction theorem does not presently exist. We record this and the other open problems, and we flag honestly that this topic is a proposed roadmap extension (QG-IX) without an assigned module.

# Contents

# 1 Introduction

## 1.1 The reconstruction thesis, and this paper's regime

The organizing thesis of this library is that quantum gravity is, in the first instance, not the quantization of material objects sitting inside a spacetime. It is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures. Different regimes of the program instantiate the same slogan with different data: a diffeomorphism-equivalence class of Einstein metrics in the classical regime, a spin network in loop quantum geometry, an isometric tensor network in holographic reconstruction, a renormalization-group fixed point in the asymptotic-safety regime. Each is a candidate answer to the question "what invariant structure, if reconstructed, would reproduce the geometry we see?"

This paper isolates the regime in which that question has its sharpest and most literal answer. In noncommutative geometry the invariant structure is a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, and the reconstruction is a theorem rather than a program: for a commutative input satisfying an explicit list of conditions, the Riemannian spin manifold is recovered from the algebra and the operator, and the recovery is unique up to the expected notion of equivalence. Nothing about a manifold, a set of points, a chart, or a metric tensor is assumed in the input. What is assumed is an algebra, a representation, and one self-adjoint operator.

The claim "geometry is an invariant of an algebra" has two layers, and it is worth separating them at the outset because they have different histories and different status. The first layer is topological and is classical: by Gelfand and Naimark, a commutative unital $C^*$-algebra $\mathcal{A}$ is the algebra $C(X)$ of continuous functions on a compact Hausdorff space $X$, and $X$ is recovered as the space of characters (pure states) of $\mathcal{A}$. The points come back from the algebra. The second layer is metric and spin-geometric, and is due to Connes: adding a single operator $D$ to the commutative algebra, and requiring the pair to satisfy the spectral-triple axioms, recovers not just the points but the geodesic distance between them, the smooth structure, the spin structure, and the Riemannian metric. The bridge between the two layers is one formula,

$$d(p,q) \;=\; \sup\big\{\, |p(a)-q(a)| \;:\; a \in \mathcal{A},\ \|[D,a]\| \le 1 \,\big\}, \tag{1}$$

Connes's distance formula, which turns a bound on the commutator with $D$ into a Lipschitz condition and reads off distance as the supremum of function differences over 1-Lipschitz functions. When $\mathcal{A} = C^\infty(M)$ and $D$ is the Dirac operator of a Riemannian spin manifold, eq. (1) returns the ordinary geodesic distance on $M$.

## 1.2 Why an operator encodes a metric

The mechanism behind eq. (1) is elementary and worth stating before any axioms. On a Riemannian manifold the Dirac operator $D$ satisfies, for a smooth function $f$ acting by multiplication,

$$[D,f] \;=\; c(df), \tag{2}$$

Clifford multiplication by the differential of $f$, whose operator norm equals the supremum of the gradient length,

$$\big\|[D,f]\big\| \;=\; \|\operatorname{grad} f\,\|_\infty \;=\; \sup_{x\in M} |df_x|. \tag{3}$$

So the constraint $\|[D, f]\| \leq 1$ is exactly the condition that $f$ is 1-Lipschitz for the geodesic distance, and the supremum in eq. (1) is the well-known dual (Monge–Kantorovich style) characterization of geodesic distance, $d(p, q) = \sup\{|f(p) - f(q)| : f \text{ 1-Lipschitz}\}$. The content of Connes's insight is that the right-hand side of this duality never mentions the points of $M$ as a manifold: it mentions only the algebra $\mathcal{A}$, the states $p, q$ on it, and the operator $D$. One may therefore *take* eq. (1) as the definition of a metric on the state space of any algebra equipped with a $D$, commutative or not, and ask when that metric reproduces a known geometry. This is the entire subject.

### 1.3 Contributions and epistemic honesty

This is an expository and synthetic paper written to a fixed house style: every substantive claim carries a status label from the project's S/H/P calculus (**[S]** standard/rigorous, **[H]** strong heuristic, **[P]** speculative; sections 2 and 5). The contributions are:

1. A self-contained account of the spectral-triple axioms, the real structure and its $KO$-dimension sign table, the distance formula, the reconstruction theorem, and the spectral action, written so that the status of each is explicit (section 2).

2. Two computations carried in full. The canonical Dirac triple of a Riemannian spin manifold, where we verify eq. (3) and hence that the distance formula returns geodesic distance (sections 3 and 4); and the two-point finite spectral triple, where we prove $d(1, 2) = 1/m$ for an off-diagonal Dirac matrix of modulus $m$ (theorem 3.4). The finite computation is checkable, and an accompanying Haskell program (section 7) evaluates the distance and confirms the formula, together with self-adjointness of $D$ and boundedness of $[D, a]$.

3. A statement of the weakest-link S/H/P calculus for this regime and its compositions with the other regimes of the library (section 5), including the honest verdict that the chain "Lorentzian reconstruction of spacetime from spectral data" is currently **[H/P]**, not **[S]** or **[H]**, because no Lorentzian analogue of the reconstruction theorem exists in the required generality.

4. An explicit statement (section 6) that this topic is a *proposed* roadmap extension, labeled QG-IX in the knowledge base, and is not among the eight named modules QG-I through QG-VIII. We do not claim it as an established module.

The honest headline is a split verdict. The kinematic core of the subject — the distance formula and the reconstruction theorem — is genuinely status **[S]**: these are theorems with proofs, in the Riemannian signature. The dynamical and phenomenological uses — a quantization of the spectral action, a path integral over spectral triples, and the detailed Standard Model predictions — are status **[H]**: strong, structurally compelling heuristics whose outputs depend on choices (of finite algebra, of cutoff function, of scale-matching scheme) not forced by the axioms. The overall dictionary label the library assigns to this topic is therefore **[H]**, and section 5 makes precise why a bare-**[H]** label sits on top of a subject containing status-**[S]** theorems.

| Status | Mathematics | Physical representation |
|---|---|---|
| **[S]** | Commutative $C^*$-algebra | set of points (Gelfand–Naimark) |
| **[S]** | Spectral triple $(\mathcal{A}, \mathcal{H}, D)$ | metric geometry encoded algebraically |
| **[S]** | Real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ | spin geometry with charge conjugation |
| **[S]** | Connes distance formula | geodesic metric from $(\mathcal{A}, D)$ |
| **[S/H]** | Reconstruction theorem | manifold recovered from algebra |
| **[S/H]** | Spectral action $\mathrm{Tr} f(D/\Lambda)$ | gravitational+matter action |
| **[H]** | Almost-commutative $C^\infty(M) \otimes \mathcal{A}_F$ | gauge and Higgs sector |
| **[H/P]** | Lorentzian/causal spectral triple | causal order from spectral data |

Table 1: Dictionary entries instantiated by the noncommutative-geometry regime, with S/H/P status. The distance formula and the reconstruction theorem are rigorous in the Riemannian setting (**[S]**); the reconstruction theorem is tagged **[S/H]** at the level of this table because its *physical* reading ("spacetime is recovered") inherits the regime's **[H]** dictionary label even though the mathematics is **[S]**. The Lorentzian row is **[H/P]**: the target theorem does not yet exist.

### 1.4 Relation to prior work

The framework is due to Connes; the standard monograph is [1] and the axioms in the form used here appear in [2, 4]. The distance formula is [1]; the reconstruction theorem is [7]. The spectral action principle is [3], and its application to the Standard Model with neutrino mixing is [6]; the modern textbook treatment is [8]. The recent survey [10] states the current status from Connes's own standpoint. The Lorentzian and causal extensions, which we treat only to delimit what is *not* yet a theorem, are [11, 12, 13, 14]. We prove nothing new about the framework; our contribution is the status-labeled synthesis, the two fully worked computations, and the checkable code, situated inside the modular reconstruction program.

A word on voice. We avoid the promotional register common in surveys of this subject. The spectral action does not "unify" gravity and the Standard Model in any sense that survives contact with the open problems in section 6; it produces both from one heat-kernel expansion, which is a real and specific fact, and we state it as such.

## 2 Mathematical framework

We reproduce, in a focused form, the translation table that opens every paper in this library. Table 1 records the dictionary entries this regime instantiates, with the project's status labels.

## 2.1 $C^*$-algebras and Gelfand–Naimark duality

We begin with the topological layer, which fixes what "recovering the points" means.

**Definition 2.1** ($C^*$-algebra)**.** A $C^*$*-algebra* is a complex Banach algebra $\mathcal{A}$ with an antilinear involution $a \mapsto a^*$ satisfying $(ab)^* = b^*a^*$ and the $C^*$-identity $\|a^*a\| = \|a\|^2$. A *character* of a commutative unital $\mathcal{A}$ is a nonzero algebra homomorphism $\chi : \mathcal{A} \to \mathbb{C}$; the set of characters, with the weak-$*$ topology, is the *spectrum* $\mathrm{Spec}(\mathcal{A})$.

**Theorem 2.2** (Gelfand–Naimark)**.** **[S]** *The functor $X \mapsto C(X)$ from compact Hausdorff spaces to commutative unital $C^*$-algebras is a contravariant equivalence of categories, with quasi-inverse $\mathcal{A} \mapsto \mathrm{Spec}(\mathcal{A})$. The evaluation map $X \to \mathrm{Spec}(C(X))$, $x \mapsto (\, f \mapsto f(x)\,)$, is a homeomorphism, and characters coincide with pure states.*

Theorem 2.2 is the prototype for everything below: a space is fully encoded by an algebra, and the space is recovered from the algebra by a canonical construction (its spectrum). What Gelfand–Naimark does *not* see is metric, smooth, or spin structure: $C(X)$ knows only the topology. The role of the operator $D$ is to supply exactly the missing metric and spin data. This is the sense in which a spectral triple is "a $C^*$-algebra with enough extra structure to be a geometry."

*Remark* 2.3. Dropping commutativity in theorem 2.1 is what makes the subject *noncommutative*: a noncommutative $C^*$-algebra has, in general, no underlying point set (its space of characters can be trivial), yet it still has states, a $K$-theory, and — once a $D$ is adjoined — a metric on its state space via eq. (1). The commutative case is the calibration point at which the constructions are required to reproduce ordinary geometry; the noncommutative case is where genuinely new "spaces" (the finite internal space $\mathcal{A}_F$ of the Standard Model, the Moyal plane, quantum tori) live.

## 2.2 Spectral triples: the axioms

We now adjoin the operator. Throughout, $\mathcal{H}$ is a separable complex Hilbert space, $\mathcal{B}(\mathcal{H})$ its bounded operators, and $\mathcal{K}(\mathcal{H})$ the compact operators.

**Definition 2.4** (Spectral triple)**.** A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of:

- a unital $*$-algebra $\mathcal{A}$ with a faithful $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ (we suppress $\pi$ and write $a$ for $\pi(a)$);
- a self-adjoint operator $D$ on $\mathcal{H}$, densely defined and generally unbounded, such that

(ST1) the resolvent $(D + i)^{-1}$ is compact, $(D + i)^{-1} \in \mathcal{K}(\mathcal{H})$;

(ST2) for every $a \in \mathcal{A}$, the operator $[D, a]$ is densely defined and extends to a bounded operator, $[D, a] \in \mathcal{B}(\mathcal{H})$.

The triple is *even* if there is a $\mathbb{Z}_2$-grading $\gamma = \gamma^*$, $\gamma^2 = 1$, with $[\gamma, a] = 0$ for all $a \in \mathcal{A}$ and $D\gamma = -\gamma D$; otherwise it is *odd*. It is *$p$-summable* ($p > 0$) if $(D + i)^{-1} \in \mathcal{L}^{p+}$, the weak Schatten ideal, equivalently if the eigenvalues $\mu_n$ of $|D|^{-1}$ satisfy $\mu_n = O(n^{-1/p})$.

Condition (ST1) is a compactness/discreteness condition: it forces $D$ to have discrete spectrum with finite-multiplicity eigenvalues accumulating only at infinity, so that the "spectrum of $D$" is a well-defined multiset of real numbers. Condition (ST2) is the Lipschitz/boundedness condition that makes eq. (1) finite on a large supply of elements. The two conditions are the algebraic distillation of "$D$ is a first-order elliptic operator on a compact manifold."

**Example 2.5** (The canonical spin triple)**.** Let $(M, g)$ be a closed Riemannian spin manifold of dimension $n$, $S \to M$ the spinor bundle, and $D$ the Dirac operator acting on $\mathcal{H} = L^2(M, S)$, with $\mathcal{A} = C^\infty(M)$ acting by multiplication. Then $(\mathcal{A}, \mathcal{H}, D)$ is an $n$-summable spectral triple, even iff $n$ is even (with $\gamma$ the chirality element of the Clifford algebra). Ellipticity of $D$ gives (ST1) via Rellich compactness; (ST2) is eq. (2). This is the triple whose reconstruction is the content of theorem 2.10.

**Example 2.6** (Finite spectral triples)**.** Let $\mathcal{A} = \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$ be a finite-dimensional $C^*$-algebra, $\mathcal{H} = \mathbb{C}^N$ finite-dimensional, and $D = D^*$ any Hermitian $N \times N$ matrix. Then $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple: (ST1) is automatic since $\mathcal{H}$ is finite-dimensional (every operator is compact) and (ST2) is automatic since every operator is bounded. These are the "0-dimensional" triples, and they carry a genuine nontrivial metric on their finite state space through eq. (1), computed in section 3. They are the building blocks of the internal space $\mathcal{A}_F$.

## 2.3 The Dirac operator and the order-one condition

For the reconstruction theorem one needs more than (ST1)–(ST2). We list the additional axioms in the form of [7, 4], stated for a commutative $\mathcal{A}$ of "metric dimension" $n$; they are the abstract counterparts of "$D$ is a Dirac-type operator on an $n$-manifold."

**Axiom 1** (Regularity/smoothness)**.** $\mathcal{A}$ and $[D, \mathcal{A}]$ lie in the smooth domain $\bigcap_{m \geq 1} \mathrm{dom}(\delta^m)$ of the derivation $\delta(T) = [|D|, T]$. This encodes that the coordinate functions are smooth, not merely Lipschitz.

**Axiom 2** (Dimension/summability and dimension spectrum)**.** $(D + i)^{-1}$ is $n$-summable and the algebra generated by $\mathcal{A}, [D, \mathcal{A}], |D|^z$ has a discrete, simple *dimension spectrum* $\Sigma \subset \mathbb{C}$. This makes the noncommutative integral $⨍ T = \mathrm{Res}_{s=0} \mathrm{Tr}(T|D|^{-s})$ well defined via the Wodzicki residue.

**Axiom 3** (Order-one condition)**.** $[[D, a], b^{\mathrm{op}}] = 0$ for all $a, b \in \mathcal{A}$, where $b^{\mathrm{op}}$ is the right action of the opposite algebra $\mathcal{A}^{\mathrm{op}}$ commuting with the left action. This is the statement that $D$ is a first-order differential operator: its commutator with a function is again "degree zero," i.e. commutes with functions acting on the other side. The distinct right action $b^{\mathrm{op}}$ is precisely the one implemented by $Jb^* J^{-1}$ once the real structure $J$ is introduced (theorem 2.7); for a bare commutative triple acting on sections, left and right multiplication coincide and the condition is trivial, so its nontrivial content is the real form $[[D, a], Jb^* J^{-1}] = 0$ of theorem 2.7.

**Axiom 4** (Orientability)**.** There is a Hochschild $n$-cycle $c = \sum a^0 \otimes a^1 \otimes \cdots \otimes a^n$ whose representative $\pi_D(c) = \sum a^0[D, a^1] \cdots [D, a^n]$ equals $\gamma$ (even case) or 1 (odd case). The cycle $c$ is the abstract volume form; this axiom fixes an orientation and the top-degree Clifford structure.

**Axiom 5** (Finiteness and absolute continuity)**.** $\mathcal{H}^\infty := \bigcap_m \mathrm{dom}(D^m)$ is a finitely generated projective $\mathcal{A}$-module, and the $\mathcal{A}$-valued inner product it carries reproduces integration against a smooth density. This makes $\mathcal{H}$ "the sections of a smooth vector bundle."

**Axiom 6** (Poincaré duality)**.** The class $[D] \in K_n(\mathcal{A})$ (in the real case, twisted by $J$) is a $K$-homological fundamental class, and cap product with it induces an isomorphism $K^*(\mathcal{A}) \to K_*(\mathcal{A})$ from $K$-theory to $K$-homology. This is the analogue of the fundamental homology class of an oriented manifold, whose cap product gives Poincaré duality between cohomology and homology.

## 2.4 Real structure and $KO$-dimension

Physical fermions require a charge conjugation, and the manifold recovered by the theorem is spin (not merely spin$^c$) precisely when a compatible real structure is present.

**Definition 2.7** (Real spectral triple)**.** A *real structure* of $KO$-dimension $k \in \mathbb{Z}/8$ on $(\mathcal{A}, \mathcal{H}, D)$ (even, with grading $\gamma$) is an antilinear isometry $J : \mathcal{H} \to \mathcal{H}$ with

$$J^2 = \varepsilon\,\mathbb{1}, \qquad JD = \varepsilon' DJ, \qquad J\gamma = \varepsilon''\gamma J, \tag{4}$$

where the signs $(\varepsilon, \varepsilon', \varepsilon'') \in \{\pm 1\}^3$ are fixed by $k$ through table 2, and such that the *order-zero condition* $[a, Jb^*J^{-1}] = 0$ and the *order-one condition* $[[D, a], Jb^*J^{-1}] = 0$ hold for all $a, b \in \mathcal{A}$. The data $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ is a *real spectral triple*.

| $k$ mod 8 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\varepsilon$ | $+$ | $+$ | $-$ | $-$ | $-$ | $-$ | $+$ | $+$ |
| $\varepsilon'$ | $+$ | $-$ | $+$ | $+$ | $+$ | $-$ | $+$ | $+$ |
| $\varepsilon''$ | $+$ | | $-$ | | $+$ | | $-$ | |

Table 2: The $KO$-dimension sign table for a real structure $J$: the signs $(\varepsilon, \varepsilon', \varepsilon'')$ of eq. (4) as a function of $k$ mod 8. Blank entries in the $\varepsilon''$ row are the odd $KO$-dimensions, where no grading $\gamma$ is present.

The operator $Jb^*J^{-1}$ implements the right multiplication by $b$; the order-zero condition says left and right multiplications commute (an $\mathcal{A}$-bimodule structure on $\mathcal{H}$), and the order-one condition is Axiom 3 in its real form. The $KO$-dimension of the Standard Model's finite triple is 6, which is the arithmetic input behind the doubling of fermionic degrees of freedom and the see-saw mechanism in [6]. We use the real structure only where needed and otherwise work with plain triples.

## 2.5 The Connes distance formula

We now state the formula precisely, at the level of states, so that it applies equally to commutative and finite triples.

**Definition 2.8** (State space and spectral distance)**.** A *state* of $\mathcal{A}$ is a positive normalized linear functional $\varphi : \mathcal{A} \to \mathbb{C}$, $\varphi(a^*a) \geq 0$, $\varphi(1) = 1$. Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the *Connes spectral distance* between states $\varphi, \psi$ is

$$d_D(\varphi, \psi) \;=\; \sup\big\{\, |\varphi(a) - \psi(a)| \;:\; a \in \mathcal{A},\; \|[D, a]\| \leq 1 \,\big\} \;\in [0, +\infty]. \tag{5}$$

For a commutative $\mathcal{A} = C(X)$, pure states are evaluations at points $x \in X$ and we write $d_D(x, y)$.

**Proposition 2.9** (Elementary properties)**.** **[S]** *The function $d_D$ is an extended pseudometric on the state space: it is symmetric, satisfies the triangle inequality, and vanishes on the diagonal. It is a genuine metric (points at finite, positive distance) precisely when $\mathcal{A}$ and $D$* separate *states in the sense that for $\varphi \neq \psi$ there is a with $\varphi(a) \neq \psi(a)$ and $\|[D, a]\| > 0$, and when $[D, a] = 0$ forces a scalar.*

*Proof.* Symmetry and the triangle inequality are immediate from the definition as a supremum of $|\varphi(a) - \psi(a)|$ over a fixed symmetric, convex balanced set of $a$'s; the diagonal vanishes because $\varphi(a) - \varphi(a) = 0$. Finiteness and positivity are the separation hypotheses. If $[D, a] = 0$ only for scalar $a$, then the constraint set is bounded modulo scalars (adding a constant to $a$ changes neither $[D, a]$ nor $\varphi(a) - \psi(a)$), so the supremum is finite; separation gives positivity for distinct states. □

The next section proves the two anchoring facts: for the canonical triple eq. (5) equals geodesic distance, and for the two-point triple it equals $1/m$.

## 2.6 The reconstruction theorem

**Theorem 2.10** (Connes reconstruction theorem, 2013)**.** **[S]** *Let $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ be a real spectral triple with $\mathcal{A}$ commutative, satisfying Axioms 1 to 6 and the KO-dimension conditions eq. (4), of metric dimension $n$. Then there is a closed oriented Riemannian spin$^c$ manifold $M$ of dimension $n$, unique up to diffeomorphism, such that $\mathcal{A} \cong C^\infty(M)$ as $*$-algebras and the triple is unitarily equivalent to the canonical Dirac triple of $M$ (theorem 2.5); the real structure upgrades spin$^c$ to spin. Under this equivalence the geodesic distance of $M$ is recovered by the distance formula eq. (5), and the Riemannian volume form is recovered by the noncommutative integral $⨍ \cdot |D|^{-n}$.*

Theorem 2.10 is the precise form of "geometry is a theorem about an algebra." The input is algebraic and operator-theoretic; the output is a manifold with all of its Riemannian and spin structure, together with a recipe (the distance formula and the noncommutative integral) recovering metric and volume. We do not reproduce the proof, which is long [7]; the mechanism of the distance part is section 3. The status is **[S]**: this is a theorem in the Riemannian signature. Section 6 states plainly that the Lorentzian analogue is open.

## 2.7 The spectral action principle

The distance formula recovers the metric; the spectral action supplies dynamics for it.

**Definition 2.11** (Spectral action)**.** Fix a cutoff scale $\Lambda > 0$ and a positive even function $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ (a smooth approximation to a cutoff). The *bosonic spectral action* of a spectral triple is

$$S_b[D] \;=\; \mathrm{Tr} f(D/\Lambda), \tag{6}$$

a spectral invariant depending only on the eigenvalues of $D$. The full action adds the *fermionic term* $S_f = \langle J\xi, D\xi\rangle$ for $\xi \in \mathcal{H}$.

**Theorem 2.12** (Heat-kernel/asymptotic expansion)**.** **[S/H]** *For the canonical triple of theorem 2.5 in dimension $n = 4$, the spectral action eq. (6) admits, as $\Lambda \to \infty$, the asymptotic expansion*

$$\mathrm{Tr} f(D/\Lambda) \;\sim\; \sum_{k\geq 0} f_{n-k}\, \Lambda^{n-k}\, a_k(D^2), \qquad f_j = \int_0^\infty f(u)\, u^{j-1}\, du \quad (j > 0), \quad f_0 = f(0), \tag{7}$$

*where $a_k(D^2)$ are the Seeley–DeWitt coefficients of $D^2$ and the moment index matches the power of $\Lambda$ (so in $n = 4$ the $k = 0$ term carries $f_4\Lambda^4$, the $k = 2$ term carries $f_2\Lambda^2$, and the $k = n = 4$ term carries $f_0 = f(0)$). Through order $\Lambda^0$ this reproduces a cosmological-constant term ($\propto \Lambda^4$), the Einstein–Hilbert action ($\propto \Lambda^2 \int R\sqrt{g}$), and Weyl-curvature-squared and Gauss–Bonnet terms ($\propto \Lambda^0$).*

The tag is **[S/H]**: the asymptotic expansion eq. (7) and the identification of the low-order coefficients are rigorous heat-kernel theory (**[S]**), but the use of $S_b[D]$ as a physical *action* to be extremized or quantized, and the reading of its coefficients as coupling constants, is a physical hypothesis (**[H]**). The almost-commutative refinement below inherits the weaker tag.

To make theorem 2.12 concrete, the relevant Seeley–DeWitt coefficients for a Laplace-type operator $D^2 = -(g^{\mu\nu}\partial_\mu\partial_\nu + \cdots) + E$ on a closed 4-manifold, written via the trace of the heat kernel $\mathrm{Tr} e^{-tD^2} \sim \sum_{k\geq 0} t^{(k-n)/2}\, a_k$, are

$$a_0 = \frac{1}{(4\pi)^2} \int_M \mathrm{Tr}(\mathbb{1})\, \sqrt{g}\, d^4x, \tag{8}$$

$$a_2 = \frac{1}{(4\pi)^2} \int_M \mathrm{Tr}\Big(\tfrac{R}{6}\mathbb{1} - E\Big) \sqrt{g}\, d^4x, \tag{9}$$

$$\begin{aligned} a_4 = \frac{1}{(4\pi)^2} \frac{1}{360} \int_M \mathrm{Tr}\Big(&12\,\Box R + 5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\ &- 60RE + 180E^2 + 30\,\Omega_{\mu\nu}\Omega^{\mu\nu} + \cdots\Big) \sqrt{g}\, d^4x, \end{aligned} \tag{10}$$

where $R$, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$ are the scalar, Ricci, and Riemann curvatures, $\Omega_{\mu\nu}$ is the curvature of the connection $\nabla$ acting on the bundle (for the fluctuated operator $D_A$ of theorem 2.16 it is the Yang–Mills field strength, and the term $30\,\Omega_{\mu\nu}\Omega^{\mu\nu}$ is the origin of the gauge kinetic action in theorem 2.13), and $E$ is the endomorphism term of the Lichnerowicz formula $D^2 = \nabla^*\nabla + \frac{R}{4}$, so that $E = \frac{R}{4}$ for the pure Dirac operator. We use the convention $D^2 = \nabla^*\nabla + E$ (endomorphism written with a plus sign), which is the negative of the Gilkey–Vassilevich endomorphism $E_{\mathrm{GV}}$ defined by $P = \nabla^*\nabla - E_{\mathrm{GV}}$; this is why $a_2$ carries $\frac{R}{6} - E$ and the $a_4$ cross term is $-60\,RE$ (both equal to the standard expressions $\frac{R}{6} + E_{\mathrm{GV}}$ and $+60\,RE_{\mathrm{GV}}$

under $E = -E_{\rm GV}$), while the $180E^2$ term is sign-independent. The total-derivative terms $12\,\Box R$ (and $60\,\Box E$, omitted in "$\cdots$") integrate to zero on the closed manifold $M$ and do not contribute to the action; we keep $12\,\Box R$ in the local integrand only to display the standard coefficient. Reading eq. (7) against eqs. (8) to (10): $a_0$ multiplies $\Lambda^4$ and is the cosmological constant (a volume term); $a_2$ multiplies $\Lambda^2$ and contains $\int R\sqrt{g}$, the Einstein–Hilbert action; and $a_4$ is the $\Lambda^0$ term carrying the higher-curvature ($R^2$, Weyl$^2$, Gauss–Bonnet) contributions. Gravity is not put in by hand; it is the $\Lambda^2$ coefficient of the eigenvalue count of $D$. That the sign and normalization of the Einstein–Hilbert term come out correctly (with a positive gravitational constant for the standard cutoff moments $f_2, f_4 > 0$) is the content that makes the spectral action a serious proposal rather than a formal manipulation.

**Theorem 2.13** (Almost-commutative spectral action, Chamseddine–Connes–Marcolli)**. [H]** *Let $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$ with $M$ a 4-manifold and $\mathcal{A}_F$ the finite algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ of the Standard Model, with $\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F$ and $D = D_M \otimes 1 + \gamma_M \otimes D_F$. Then the asymptotic expansion of $\operatorname{Tr} f(D/\Lambda)$ reproduces the Einstein–Hilbert action together with the full bosonic sector of the Standard Model — the $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ Yang–Mills action and the Higgs potential — with specific relations among the gauge couplings and the Higgs self-coupling imposed at the scale $\Lambda$.*

The relations of theorem 2.13 (e.g. $g_2^2 = g_3^2 = \frac{5}{3} g_1^2$ at $\Lambda$, and a Higgs-mass constraint) are the phenomenological content, and they are the reason the topic-level dictionary label is **[H]**: they depend on the choice of $\mathcal{A}_F$, on $f$ through the moments $f_0, f_2, f_4$, and on the assumption that the relations hold at a unification scale and run down. The choice of $\mathcal{A}_F$ is constrained but not uniquely forced by the axioms (section 6).

## 2.8 The noncommutative integral, Weyl's law, and the Dixmier trace

The spectral action recovered curvature integrals; the tool that makes "integral" a spectral notion is the Dixmier trace, and it is worth stating because it closes the loop between "the spectrum of $D$" and "the volume of $M$" that theorem 3.6(ii) illustrated in one dimension.

**Definition 2.14** (Noncommutative integral)**.** For an $n$-summable spectral triple, the *noncommutative integral* of $T \in \mathcal{B}(\mathcal{H})$ is

$$⨍ T \;:=\; \operatorname{Tr}_\omega\big(T\,|D|^{-n}\big), \tag{11}$$

where $\operatorname{Tr}_\omega$ is the Dixmier trace, the singular trace that measures the logarithmic divergence of the ordinary trace on the ideal $\mathcal{L}^{1+} = \{S : \sum_{k<N} \mu_k(S) = O(\log N)\}$ of eigenvalue sums; on this ideal $\operatorname{Tr}_\omega(S) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{k<N} \mu_k(S)$ when the limit exists.

**Proposition 2.15** (Connes trace theorem, dimension and volume)**. [S]** *For the canonical triple of theorem 2.5, the eigenvalue counting function of $|D|$ obeys Weyl's law $N(\Lambda) \sim \dfrac{\operatorname{rank}(S)\,\Omega_n\ \operatorname{vol}(M)}{(2\pi)^n}\,\Lambda^n$, where $\Omega_n$ is the volume of the unit ball and $\operatorname{rank}(S) = 2^{\lfloor n/2 \rfloor}$ is the* *rank of the spinor bundle (the eigenvalues of $|D|$ carry this fiber multiplicity), so that the* dimension $n$ *is the growth exponent of the spectrum and the* volume *is*

$$\operatorname{vol}(M) \;=\; \frac{(2\pi)^n}{\operatorname{rank}(S)\,\Omega_n}\,\lim_{\Lambda\to\infty}\frac{N(\Lambda)}{\Lambda^n} \;=\; c_n \fint \mathbb{1}, \tag{12}$$

*with $c_n$ a universal constant that absorbs* $\operatorname{rank}(S)$*. More generally $\fint f = c_n \int_M f \operatorname{vol}_g$ for $f \in C^\infty(M)$: the noncommutative integral is, up to normalization, Riemannian integration.*

*Sketch.* $|D|^{-n}$ lies in $\mathcal{L}^{1+}$ precisely because $D$ is $n$-summable, and the Dixmier trace of $|D|^{-n}$ is computed from the leading Weyl asymptotics of the eigenvalues, which for a first-order elliptic operator on $(M, g)$ are fixed by the principal symbol and hence by the metric. The identification of the resulting local functional with $\int_M \operatorname{vol}_g$ is Connes's trace theorem, which equates the Dixmier trace of a pseudodifferential operator of order $-n$ with its Wodzicki residue, a local integral of the symbol. Applying this to $f|D|^{-n}$ gives $\fint f \propto \int_M f \operatorname{vol}_g$. □

Theorem 2.15 is the sense in which *all* of Riemannian integration — dimension, volume, and integration of functions — is read off from the spectrum of $D$. Combined with theorem 3.2 (distance from $D$), it shows that the two pieces of Riemannian data a physicist cares about, lengths and volumes, are both spectral invariants. This is the quantitative backbone of theorem 2.10.

## 2.9 Inner fluctuations: how gauge fields appear

One more construction is needed to see why the almost-commutative geometry produces gauge *fields* and not merely gauge *groups*. Morita equivalence of the algebra, implemented on the same Hilbert space, replaces $D$ by a fluctuated operator.

**Definition 2.16** (Inner fluctuation)**.** Given a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$, a *gauge potential* is a self-adjoint element $A = \sum_i a_i\,[D, b_i]$ with $a_i, b_i \in \mathcal{A}$, $A = A^*$. The *inner fluctuation* of $D$ is

$$D_A \;=\; D + A + \varepsilon'\, JAJ^{-1}, \tag{13}$$

which is again self-adjoint and defines a spectral triple $(\mathcal{A}, \mathcal{H}, D_A, J, \gamma)$ with the same $K$-homology class.

For the canonical manifold triple the fluctuations $A = \sum a_i[D, b_i] = \sum a_i c(db_i) = c(\alpha)$ range over Clifford multiplications by real 1-forms $\alpha$, which are exactly $U(1)$ gauge potentials. For the almost-commutative triple $C^\infty(M) \otimes \mathcal{A}_F$ the fluctuations split into a continuous part (the 1-form gauge fields of the group $\mathcal{A}_F^\times$, giving the $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ gauge bosons) and a discrete part (the off-diagonal fluctuations of $D_F$, giving the Higgs field, as in section 4.2). The spectral action $\operatorname{Tr} f(D_A/\Lambda)$ evaluated on the fluctuated operator produces the Yang–Mills and Higgs kinetic and potential terms [6, 8]. The gauge fields are, in this language, coordinates on the space of geometries Morita-equivalent to the starting one: the same statement as "the metric is $D$," applied to the internal directions. We label the identification with physical gauge fields **[H]**, consistent with the rest of the almost-commutative construction.

# 3 Main results: two distance computations

This section proves the two facts that anchor the framework: that the distance formula returns geodesic distance in the canonical case, and that it returns $1/m$ for the two-point space. The second is the property checked by the accompanying code.

## 3.1 The gradient–commutator identity and geodesic distance

**Lemma 3.1** (Commutator equals Clifford action of the differential)**.** **[S]** *Let $D$ be the Dirac operator of a Riemannian spin manifold $(M, g)$ acting on $L^2(M, S)$, and let $f \in C^\infty(M)$ act by multiplication. Then*

$$[D, f] \;=\; c(df), \tag{14}$$

*Clifford multiplication by the 1-form $df$, and consequently*

$$\big\|[D, f]\big\| \;=\; \sup_{x\in M} |df_x|_g \;=\; \mathrm{Lip}(f), \tag{15}$$

*the Lipschitz constant of $f$ for the geodesic distance $d_g$.*

*Proof.* Locally $D = \sum_j c(e^j)\, \nabla^S_{e_j}$ for an orthonormal coframe $\{e^j\}$ and the spin connection $\nabla^S$. Since $\nabla^S$ is a connection and $f$ is a scalar, $\nabla^S_{e_j}(f\psi) = (e_j f)\psi + f\nabla^S_{e_j}\psi$, so $[D, f]\psi = \sum_j c(e^j)(e_j f)\psi = c\big(\sum_j (e_j f) e^j\big)\psi = c(df)\psi$. For the norm, Clifford multiplication by a 1-form $\alpha$ satisfies $c(\alpha)^* c(\alpha) = |\alpha|_g^2\, \mathbb{1}$ pointwise (the Clifford relation $c(\alpha)^2 = -|\alpha|^2$ together with $c(\alpha)^* = -c(\alpha)$), so the fiberwise operator norm of $c(df)_x$ is $|df_x|_g$, and the operator norm on $L^2$ is the supremum over $x$. Finally $\sup_x |df_x|_g = \mathrm{Lip}(f)$ because the length of the gradient bounds the rate of change of $f$ along unit-speed geodesics and is attained. □

**Theorem 3.2** (Distance formula recovers geodesic distance)**.** **[S]** *For the canonical triple of theorem 2.5 and points $x, y \in M$,*

$$d_D(x, y) \;=\; d_g(x, y), \tag{16}$$

*the Riemannian geodesic distance.*

*Proof.* By theorem 3.1 the constraint $\|[D, f]\| \le 1$ is exactly $\mathrm{Lip}(f) \le 1$. Hence

$$d_D(x, y) = \sup\{|f(x) - f(y)| : \mathrm{Lip}(f) \le 1\}.$$

The right-hand side is the Kantorovich–Rubinstein dual expression for $d_g$: for the 1-Lipschitz function $f(z) = d_g(z, y)$ one has $|f(x) - f(y)| = d_g(x, y)$, giving "$\ge$", and for any 1-Lipschitz $f$, $|f(x) - f(y)| \le d_g(x, y)$ by definition of the Lipschitz constant, giving "$\le$". (Smoothness of the competitors is not a loss: $d_g(\cdot, y)$ is 1-Lipschitz and can be uniformly approximated by smooth 1-Lipschitz functions, and the supremum is unchanged.) □

Theorem 3.2 is the calibration: on the input where the answer is known, eq. (5) returns the correct metric. It is the distance half of theorem 2.10, isolated and proved directly.

### 3.2 The two-point space

We now compute the metric of the simplest genuinely finite spectral triple. This is the worked example whose numeric value the code checks.

**Definition 3.3** (Two-point spectral triple)**.** Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C} \cong C(\{1, 2\})$, acting on $\mathcal{H} = \mathbb{C}^2$ by $\pi(a_1, a_2) = \operatorname{diag}(a_1, a_2)$. Fix $m \in \mathbb{C}^\times$ and set

$$D \;=\; \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix} \;=\; D^*. \tag{17}$$

Then $(\mathcal{A}, \mathcal{H}, D)$ is a finite spectral triple (theorem 2.6). Its two pure states are $p_1(a) = a_1$ and $p_2(a) = a_2$.

**Theorem 3.4** (Two-point distance)**. [S]** *For the two-point spectral triple of theorem 3.3,*

$$d_D(p_1, p_2) \;=\; \frac{1}{|m|}. \tag{18}$$

*Proof.* Write $a = (a_1, a_2)$, so $\pi(a) = \operatorname{diag}(a_1, a_2)$. Compute the commutator:

$$[D, \pi(a)] = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix} = \begin{pmatrix} 0 & m(a_2 - a_1) \\ \bar{m}(a_1 - a_2) & 0 \end{pmatrix}.$$

This is an off-diagonal matrix $M = \left(\begin{smallmatrix} 0 & b \\ c & 0 \end{smallmatrix}\right)$ with $b = m(a_2 - a_1)$ and $c = \bar{m}(a_1 - a_2)$. For *any* complex $b, c$ one has $MM^* = \operatorname{diag}(|b|^2, |c|^2)$, so the operator norm is $\max(|b|, |c|)$. Here $|b| = |c| = |m|\,|a_1 - a_2|$ (no reality assumption on $a$ is needed, only the moduli), hence

$$\big\|[D, \pi(a)]\big\| \;=\; |m|\,|a_1 - a_2|. \tag{19}$$

The constraint $\|[D, \pi(a)]\| \le 1$ is therefore $|a_1 - a_2| \le 1/|m|$, and

$$d_D(p_1, p_2) = \sup\big\{\, |p_1(a) - p_2(a)| : \|[D, \pi(a)]\| \le 1 \,\big\} = \sup\big\{\, |a_1 - a_2| : |a_1 - a_2| \le \tfrac{1}{|m|} \,\big\} = \frac{1}{|m|},$$

where the middle equality uses $p_1(a) - p_2(a) = a_1 - a_2$, and the supremum is attained at any $a$ with $|a_1 - a_2| = 1/|m|$. $\square$

Two features of theorem 3.4 deserve comment. First, the distance is *inversely* proportional to the off-diagonal Dirac entry: a large $m$ (strong coupling between the two points in $D$) makes the points *close*. This is the finite shadow of eq. (15): $D$ plays the role of an inverse length scale. Second, the two points are at finite distance despite there being no path between them — the metric is defined purely algebraically, with no notion of curve. This is exactly what lets the same formula produce a metric on the internal space $\mathcal{A}_F$, where the "distance" between the two sheets is set by the Higgs vacuum expectation value entering $D_F$; the Higgs field is, in this language, the fluctuation of the finite metric [6, 8].

**Corollary 3.5** (Recovering a metric quantity from $D$)**. [S]** *The single spectral datum $|m| = \|D\|$ of the two-point triple is recovered from the metric it defines by $|m| = 1/d_D(p_1, p_2)$. Equivalently, the map $m \mapsto d_D$ is a bijection between positive off-diagonal Dirac data and two-point metrics, exhibiting "the geometry is the operator" in the smallest nontrivial case.*

### 3.3 A one-dimensional continuum check: the circle

To connect the finite computation back to the continuum, we record the circle, where the spectrum of $D$ alone recovers a metric invariant (the circumference) through Weyl's law, and the distance formula recovers arc length.

**Proposition 3.6** (The circle triple). **[S]** *Let $M = S^1$ of circumference $L$, with arc-length coordinate $s \in \mathbb{R}/L\mathbb{Z}$, $\mathcal{H} = L^2(S^1)$, $\mathcal{A} = C^\infty(S^1)$, and $D = -i\,\frac{d}{ds}$ for the trivial (periodic) spin structure. Then:*

*(i)* $\mathrm{Spec}(D) = \frac{2\pi}{L}\mathbb{Z}$*, each eigenvalue simple, with eigenfunctions* $e^{2\pi iks/L}$*;*

*(ii) the eigenvalue counting function obeys Weyl's law* $N(\Lambda) := \#\{\lambda \in \mathrm{Spec}(D) : |\lambda| \leq \Lambda\} \sim \frac{L}{\pi}\Lambda$*, so $L$ is recovered from the spectrum by* $L = \pi \lim_{\Lambda\to\infty} N(\Lambda)/\Lambda$*;*

*(iii) the distance formula gives* $d_D(s, s') = \min(|s - s'|, L - |s - s'|)$*, the geodesic (arc-length) distance.*

*Proof.* (i) $De^{2\pi iks/L} = \frac{2\pi k}{L}e^{2\pi iks/L}$, and these functions are a Hilbert basis. (ii) The number of integers $k$ with $|2\pi k/L| \leq \Lambda$ is $2\lfloor L\Lambda/2\pi\rfloor + 1 \sim \frac{L}{\pi}\Lambda$. (iii) $[D, f] = -if'$ acts by multiplication, so $\|[D, f]\| = \|f'\|_\infty$, and $\|f'\|_\infty \leq 1$ is 1-Lipschitzness for arc length; theorem 3.2's argument applied to $S^1$ gives the stated distance, the geodesic distance being the shorter of the two arcs. (For the non-trivial, bounding spin structure the eigenvalues shift to $\frac{2\pi}{L}(k + \frac{1}{2})$; the Weyl asymptotics in (ii), and hence the recovered circumference, are unaffected.) □

Part (ii) is a small but complete instance of the reconstruction philosophy for a metric *scalar*: no coordinate on $S^1$ is used; the circumference is read off from the asymptotic density of eigenvalues of $D$. In dimension $n$, Weyl's law $N(\Lambda) \sim c_n \,\mathrm{vol}(M)\Lambda^n$ recovers both the dimension $n$ (from the growth exponent) and the volume, which is the spectral origin of Axiom 2 and of the noncommutative integral in theorem 2.10.

# 4 Worked examples in coordinates

We collect the three examples of section 3 into a comparison and add the almost-commutative two-sheet model, to make the "Higgs as finite metric" statement concrete.

### 4.1 Comparison table

### 4.2 The two-sheet (almost-commutative) model

Take $M$ a Riemannian 4-manifold and $\mathcal{A}_F = \mathbb{C}\oplus\mathbb{C}$, so $\mathcal{A} = C^\infty(M)\otimes\mathcal{A}_F = C^\infty(M)\oplus C^\infty(M)$: two copies of $M$, a "two-sheeted" space. Let $\mathcal{H} = L^2(M, S)\otimes\mathbb{C}^2$ and

$$D \;=\; D_M \otimes \mathbb{1}_2 \;+\; \gamma_M \otimes \begin{pmatrix} 0 & \phi \\ \bar{\phi} & 0 \end{pmatrix}, \tag{20}$$

with $\phi$ a (for now constant) complex parameter and $\gamma_M$ the chirality grading. The state space is two copies of $M$, and the distance between corresponding points $(x, 1)$ and $(x, 2)$ on the two

| Triple | Algebra $\mathcal{A}$ | Operator $D$ | Metric recovered |
|---|---|---|---|
| Spin manifold | $C^\infty(M)$ | Dirac operator | geodesic $d_g(x,y)$ |
| Circle $S^1$ | $C^\infty(S^1)$ | $-i\,d/ds$ | arc length; $L$ via Weyl |
| Two points | $\mathbb{C}\oplus\mathbb{C}$ | $\left(\begin{smallmatrix}0 & m\\ \bar{m} & 0\end{smallmatrix}\right)$ | $1/\lvert m\rvert$ |
| Two sheets | $C^\infty(M)\otimes(\mathbb{C}\oplus\mathbb{C})$ | $D_M\otimes 1+\gamma\otimes D_F$ | fiber distance $\propto 1/\phi$ |

Table 3: The four calibration examples. In each case the metric on the (pure) state space is produced by eq. (5) from the pair $(\mathcal{A}, D)$ alone. The last row is the almost-commutative model of section 4.2, in which the finite off-diagonal datum is a Higgs field $\phi$.

sheets is, by the same computation as theorem 3.4 applied fiberwise, $d_D((x,1),(x,2)) = 1/|\phi|$, while the distance along each sheet is the geodesic distance of $M$. The geometry is a pair of copies of $M$ held apart by a "distance in the fifth, discrete direction" equal to $1/|\phi|$.

Promoting $\phi$ to a field $\phi(x)$ — an inner fluctuation of $D$ of the form $D \mapsto D + A + JAJ^{-1}$ with $A = \sum a_i[D, b_i]$ — turns the constant off-diagonal entry into a scalar field on $M$ with exactly the quantum numbers of the Higgs, and the spectral action eq. (6) evaluated on eq. (20) produces the Higgs kinetic term and a quartic potential [3, 6, 8]. This is the precise sense of "the Higgs field is the metric of the internal space": it is the off-diagonal Dirac datum whose modulus sets the distance between the two sheets. We label the physical identification **[H]**: the mathematics of eq. (20) and its spectral action is rigorous heat-kernel computation, but the identification of $\phi$ with the physical Higgs, and the resulting mass relation, are model dependent.

# 5 Relation to other regimes and the weakest-link status calculus

The library is modular, not unified: its regimes are chained by a monotone composition of epistemic warrants, never merged into a single structure. This section states the calculus and applies it to the compositions in which the spectral regime participates.

## 5.1 The status calculus

**Definition 5.1** (Warrants and composition)**.** Let warrants be ordered **[S]** < **[H]** < **[P]** (standard, heuristic, speculative). A *status* is a nonempty set of warrants written **[S]**, **[H]**, **[P]**, or composite **[S/H]**, **[H/P]**; its *worst* component is the maximum under the order. For a chain of representations with statuses $\sigma_1, \ldots, \sigma_k$, the composite status is

$$\mathrm{compose}(\sigma_1, \ldots, \sigma_k) \;=\; \max_i \;\mathrm{worst}(\sigma_i), \tag{21}$$

the worst warrant appearing anywhere in the chain.

**Proposition 5.2** (Monotonicity and the unit law). **[S]** *Composition eq.* (21) *is monotone: for all* $\sigma, \tau$*,* $\mathrm{worst}(\mathrm{compose}(\sigma, \tau)) \geq \mathrm{worst}(\sigma)$ *and* $\geq \mathrm{worst}(\tau)$*. The warrant* **[S]** *is a unit:* $\mathrm{compose}(\textbf{[S]}, \sigma) = \mathrm{worst}(\sigma)$*. Consequently no chain can be rated* above *its weakest link, and inserting a rigorous (***[S]***) step never changes a composite's status.*

*Proof.* Both statements are properties of max on a totally ordered set: max dominates each argument (monotonicity), and the minimum element **[S]** is the identity for max (unit). These are the executable facts checked in section 7. ☐

*Remark* 5.3. Theorem 5.2 explains the apparent tension in table 1: the subject contains status-**[S]** theorems (theorems 2.10, 3.2 and 3.4) yet the topic's dictionary label is **[H]**. The label is the status of the intended *physical* use — a dynamical, quantum-gravitational reconstruction of spacetime — which chains the rigorous kinematics (**[S]**) with a dynamical hypothesis (a quantized spectral action, **[H]**). By eq. (21) the composite is **[H]**. The rigor of the kinematics is not lost; it is that the physical claim routes through a heuristic step, and the calculus reports the weakest link.

## 5.2 Compositions with the other regimes

**With the classical regime (lorentzian-geometry, QG-I).** The classical regime's invariant is the diffeomorphism-equivalence class of Einstein metrics, packaged as the solution stack $\mathrm{Sol(Ein)}/\!\!/\mathrm{Diff}$, with status **[S/H]**. The spectral regime offers a different presentation of the *same* metric datum: a Riemannian metric is equivalently a canonical spectral triple (theorem 2.10), and this equivalence of presentations is a status-**[S]** theorem. Composing the classical regime's dictionary status **[S/H]** with this **[S]** bridge gives, by the worst-component-wins rule eq. (21), $\mathrm{worst}\big(\mathrm{compose}(\textbf{[S/H]}, \textbf{[S]})\big) = \textbf{[H]}$: the mathematical bridge is rigorous, but read as a physical statement the composite still carries the **[H]** component of the classical regime's **[S/H]** label. The physically desired composition, however, is the *Lorentzian* one:

**Proposition 5.4** (The Lorentzian reconstruction chain is H/P). **[H/P]** *Let* $\sigma_1 = \textbf{[S/H]}$ *be the status of the classical Lorentzian regime and* $\sigma_2$ *the dictionary status* **[H]** *of "recover a Lorentzian causal metric from spectral/algebraic data." The calculus eq.* (21) *gives* $\mathrm{worst}\big(\mathrm{compose}(\sigma_1, \sigma_2)\big) = \textbf{[H]}$*. This is optimistic: since no Lorentzian analogue of theorem 2.10 exists in the required generality (section 6), the general claim is not established even heuristically, and the honest label bumps to* **[H/P]** *—* **[H]** *where partial constructions exist (Krein-space and causal-cone formulations [11, 14, 13]),* **[P]** *for the general reconstruction. We therefore record the chain as* **[H/P]**.

This is the sharpest honest statement the calculus produces about this regime: the Riemannian reconstruction is a theorem, but the Lorentzian reconstruction that the physics actually needs is not, and the composite status reflects that gap rather than papering over it.

**With the discrete-causal regime (causal sets, proposed QG-X).** The causal-set program reconstructs a Lorentzian manifold from an order-plus-volume datum, conjecturally (the "Hauptvermutung"), status **[H]**/**[P]**. It shares the reconstruction logic with this regime: algebraic/order data first, manifold as a theorem or conjecture second. Composing spectral and causal reconstructions of the *same* Lorentzian target keeps the **[P]** component of each, $\operatorname{worst}\big(\operatorname{compose}(\textbf{[H/P]}, \textbf{[H/P]})\big) = \textbf{[P]}$; the two are complementary attacks on the one missing Lorentzian theorem, not independent confirmations of it, and the composite honestly reads **[H/P]**.

**With factorization/observable regimes.** The noncommutative integral $⨍ \cdot |D|^{-n}$ and the spectral action provide an *observable* functional on the spectral datum, in the sense of the library's realization pipeline $M \to \Phi(M) \to \mathrm{Real}(\Phi(M)) \to \mathrm{Obs}$: the algebra and $D$ are $\Phi(M)$, the trace is the realization, and $S_b[D]$ is the observable. This places the spectral regime inside the same pipeline formalism as the BV/factorization and amplitude regimes, at status **[S/H]** for the integral (rigorous, via the Wodzicki residue) and **[H]** for its use as a physical action.

# 6 Limitations and open problems

We state the limitations plainly, in decreasing order of severity for the reconstruction thesis.

**Signature: no Lorentzian reconstruction theorem.** theorem 2.10 and theorem 3.2 are Riemannian. Their proofs use the compact-resolvent axiom (ST1) and self-adjointness of $D$, both tailored to elliptic operators in Euclidean signature. Physical spacetime is Lorentzian, where the Dirac operator is hyperbolic, the natural inner product is indefinite (a Krein rather than Hilbert space), and "compact resolvent" fails. Several partial frameworks exist — Krein spectral triples [14], temporal Lorentzian spectral triples [12], the causal-cone formulation [11], and pseudo-Riemannian axiom sets [13] — but there is at present no Lorentzian analogue of the reconstruction theorem recovering a causal Lorentzian manifold from spectral/algebraic data in the generality of the Riemannian result. We label the general Lorentzian reconstruction claim **[H/P]** and identify it as the primary open problem of the regime, directly overlapping the causal-set reconstruction of the QG-X proposal.

**Dynamics: no quantization of the spectral action.** The spectral action eq. (6) is a classical/semiclassical functional; its heat-kernel expansion is asymptotic, not a convergent quantization. There is no established, well-defined path integral over spectral triples — no measure on the space of Dirac operators or of triples — so the framework supplies kinematics (which geometries exist, and their metric content) but not an independent quantum-gravitational dynamics. Any claim of the form "quantum gravity is a sum over spectral triples" is **[P]**.

**Non-uniqueness of the finite algebra and cutoff dependence.** The finite algebra $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ of the Standard Model is strongly constrained by the axioms (and by classification results of Chamseddine, Connes, and van Suijlekom on irreducible finite geometries), but it is not *uniquely* forced: variants exist, and the phenomenological outputs of theorem 2.13 depend on the choice of $\mathcal{A}_F$, on the cutoff function $f$ through its moments, and on the assumption that the coupling relations hold at a unification scale. The Higgs-mass "prediction" of the 2007 model, taken at face value, sits above the measured value and requires subsequent refinements (an additional scalar field $\sigma$). These are the reasons the phenomenology is **[H]**, not **[S]**.

**Status as a proposed roadmap extension.** Honesty about scope: this topic is *not* one of the eight named modules QG-I through QG-VIII of the project roadmap. The knowledge base proposes it as an extension, QG-IX (spectral/algebraic reconstruction), precisely because the dictionary already contains the entry "spectral triple / noncommutative algebra" at status **[H]** but no roadmap module carried it. We present the regime as this proposed extension and do not claim it as an established module. The companion proposal QG-X (causal-order reconstruction) is its natural partner, and the two share the open Lorentzian-reconstruction problem above.

# 7 A machine-checkable model

The finite computations of this paper are small enough to verify by direct evaluation. The accompanying Haskell package `src/noncommutative-geometry/` provides:

- `Core.hs`: complex $2 \times 2$ matrices, the two-point Dirac operator eq. (17), the operator norm of a $2 \times 2$ matrix via its singular values, the commutator $[D, \pi(a)]$, and the two-point Connes distance computed by maximizing $|a_1 - a_2|$ subject to $\|[D, \pi(a)]\| \leq 1$ (in closed form, and by a sampling check).
- `Status.hs`: the S/H/P warrant lattice of theorem 5.1 with the `compose` operation, and property checks for monotonicity and the unit law (theorem 5.2).
- `Main.hs`: a runnable demonstration that prints the two-point distance for several values of $m$ and checks it against $1/|m|$, verifies self-adjointness of $D$ and boundedness of $[D, \pi(a)]$, confirms eq. (19), and runs the status-calculus properties.

The load-bearing checkable property is theorem 3.4: the program confirms

$$\left| d_D(p_1, p_2) - 1/|m| \right| < 10^{-9} \qquad \text{for } m \in \{1,\ 2,\ \tfrac{1}{2},\ 3 + 4i\}, \tag{22}$$

using both the closed-form value and an independent constrained-maximization sampler, so the two agree. The build is base-only Haskell and compiles with `ghc -o demo Main.hs Core.hs Status.hs`.

# 8 Conclusion

The spectral-triple regime is the most literal instance in this library of the reconstruction thesis. In it, "spacetime is reconstructed from invariant representation structures" is not a program but a theorem: Gelfand–Naimark recovers the points of a space from a commutative $C^*$-algebra; the Connes distance formula eq. (5) recovers the geodesic metric from the pair $(\mathcal{A}, D)$ (theorem 3.2); and Connes's reconstruction theorem (theorem 2.10) recovers the entire Riemannian spin manifold, uniquely up to diffeomorphism, from a commutative real spectral triple satisfying an explicit axiom list. The two computations we carried in full — geodesic distance for the canonical triple, and $d_D = 1/|m|$ for the two-point triple (theorem 3.4, checked by code) — exhibit the mechanism at both ends of the dimension range.

The honest verdict is a split one, and the S/H/P calculus is the instrument that keeps it honest. The kinematic core is status **[S]**: these are theorems, in Riemannian signature. The dynamical and phenomenological uses are status **[H]**, and the specific reconstruction the physics needs — a Lorentzian one, recovering causal structure rather than only a Riemannian metric — is status **[H/P]**, because the required theorem does not yet exist. The topic is presented as a proposed roadmap extension, QG-IX, not as an established module, and its central open problem is shared with its proposed partner QG-X. What the regime establishes with full rigor is a proof of concept for the entire library's thesis: there is at least one physically relevant category of geometry that is, provably, an invariant of algebraic representation data and nothing else.

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