The Solution Stack of General Relativity: Diffeomorphism Equivalence, Isotropy, and the Classical Geometry Regime of a Reconstruction Program

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Abstract

We treat the classical regime of general relativity as one layer of a broader program in which spacetime, matter, and causality are reconstructed from invariant representation structures rather than obtained by quantizing material fields on a fixed background. In this regime the invariant is diffeomorphism equivalence, and we argue that the correct carrier of its physical content is the action groupoid $\mathrm{Sol}(\mathrm{Ein}) /\!\!/ \mathrm{Diff}(M)$ — the moduli *stack* of Einstein solutions — and not the coarse orbit set $\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)$. The distinction is not cosmetic. We collect the classical facts that force it: the Ebin slice theorem identifies the isotropy group of the $\mathrm{Diff}(M)$-action at a metric g with the isometry group $\mathrm{Isom}(M, g)$, and the linearization-stability theorems of Moncrief and of Fischer, Marsden, Arms and Moncrief show that the orbit space acquires quadratic (conical) singularities at exactly those solutions that carry Killing fields. The stack is smooth where the coarse quotient is singular because it records $\mathrm{Aut}_{[\mathrm{Sol}/\mathrm{Diff}]}([g]) \cong \mathrm{Isom}(M, g)$ as automorphism data rather than collapsing it. We give the moduli stack its groupoid definition, prove the isotropy identification, state the deformation complex whose zeroth cohomology is the Killing algebra and whose first cohomology is on-shell perturbations modulo gauge, and compute isotropy for the Minkowski, Schwarzschild, Kerr and Friedmann–Lemaître–Robertson–Walker families, exhibiting the isotropy jump $\mathrm{Isom} = \mathbb{R} \times U(1) \rightsquigarrow \mathbb{R} \times SO(3)$ along the Kerr-to-Schwarzschild limit as an explicit stratification of the stack. Each result carries an explicit epistemic-status label (standard mathematics S, physical heuristic H, speculative P), and we place the regime inside a modular, weakest-link status calculus that governs how it composes with eleven other regimes of the same program without ever asserting a monolithic unification. We state honestly what is not settled: a global four-dimensional derived stack of solutions, with non-proper Diff and matter, is not available in the generality the program would like.

Contents

1 Introduction

1.1 The reconstruction thesis, and this paper's regime

The organizing perspective of the wider project this paper belongs to can be stated in one sentence. *Quantum gravity is not primarily the quantization of material objects in spacetime; it is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures.* Under that reading, a classical spacetime is not a substrate one starts from and then quantizes. It is an output: the semiclassical shadow that any deeper reconstruction — a spin network, a tensor-network holographic code, a celestial correlator, an asymptotically safe renormalization-group trajectory, a spectral triple, a causal set — must reproduce in an appropriate limit. If that is the role of classical general relativity, then the classical theory has one job in the program: to package its own content in a form precise enough that "reproduce the classical limit" becomes a checkable statement rather than a slogan.

This paper carries out that packaging for the classical and semiclassical geometry regime. The invariant here is diffeomorphism equivalence: two Lorentzian metrics related by a diffeomorphism of the underlying manifold describe the same physical gravitational configuration, because the diffeomorphism is a relabelling of coordinates and general covariance forbids coordinates from carrying physical content. The naive way to encode this is to pass to the set of orbits, the coarse quotient $\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)$ of Einstein solutions modulo diffeomorphism. Our central claim is that this is the wrong object, and that the right one is the action groupoid, written $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$ and read as a stack: its objects are solutions, its morphisms are diffeomorphisms carrying one solution to another, and — this is the point — its automorphism group at a solution g is the isometry group $\mathrm{Isom}(M, g)$.

1.2 Why the coarse quotient is not enough

The reason is a theorem, not a preference. Two classical results, both from the 1970s and 1980s and both entirely rigorous, converge on it.

The first is the slice theorem of Ebin [8] (with the infinite-dimensional manifold structure developed by Fischer and Marsden [9]). Near a metric g the action of $\mathrm{Diff}(M)$ on the space of metrics admits a slice, and the local model for the orbit space is (slice)/$\mathrm{Isom}(M, g)$. The isotropy group of the action at g is exactly the isometry group. When $\mathrm{Isom}(M, g)$ is trivial the orbit space is locally a manifold; when it is not, the orbit space is locally a quotient of a linear slice by a nontrivial compact-or-noncompact group, and it fails to be a manifold there.

The second is linearization stability. Moncrief [10, 11] and then Fischer, Marsden, Arms and Moncrief [12, 13] proved that for a vacuum spacetime with a compact constant-mean-curvature Cauchy surface, the space of solutions is a smooth manifold except at solutions admitting Killing fields, where it has a quadratic singularity governed by the Killing algebra. Equivalently, such a spacetime is linearization stable if and only if it admits no global Killing vector field. The singular points of $\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)$ are precisely the symmetric solutions.

Put those together and the situation is unambiguous. The coarse orbit space is singular exactly where isotropy is nontrivial, and it is singular there because it has thrown away the isotropy. The stack keeps the isotropy as automorphism data. It is smooth — in the appropriate stacky sense — at the points where the coarse quotient tears, and the information it retains at those points, the Killing algebra of the solution, is exactly the data that has direct physical meaning: conserved Noether charges, the reducibility of the gravitational gauge algebra, and, once the classical regime is reconstructed as a limit of a deeper theory, the residual symmetry that the deeper theory must match.

1.3 Contributions and epistemic honesty

We do three things. First (sections 2 and 3) we give precise definitions of the Lorentzian, causal, and Einstein data, and of the solution stack as an action groupoid, and we prove the isotropy identification $\mathrm{Aut}_{[\mathrm{Sol}/\mathrm{Diff}]}([g]) \cong \mathrm{Isom}(M, g)$ together with the statement that the coarse quotient's singular locus is the symmetric locus. Second (sections 4 and 5) we compute isotropy for the standard solution families and exhibit the isotropy jump along the Kerr-to-Schwarzschild limit as an explicit stratification, and we write down the deformation complex whose cohomology detects it. Third (sections 6 and 7) we record the connection reformulation that bridges this regime to the spin-network regime, and we place the whole regime inside a modular weakest-link status calculus.

Throughout, every substantive claim carries an epistemic-status label. We use three ordered warrants: S for standard mathematics or established mathematical physics, H for a strong physical heuristic, and P for a speculative ontological extension, with $\mathsf{S} < \mathsf{H} < \mathsf{P}$. A label such as **[S/H]** means the claim's warrant sits between the two. The rule that governs composition of labelled claims is that the composite inherits the worst (largest) warrant in the chain; we make this precise in section 7 and use it to bound, honestly, what can be asserted when this regime is chained to the others. We also state plainly, in section 9, what is not settled: the global four-dimensional derived stack of solutions, with non-proper diffeomorphism action and matter coupling, is not constructed in the literature at the generality the program would want, and we do not pretend otherwise.

1.4 Relation to prior work

The idea that the configuration space of gravity is a quotient stack $\mathrm{Met}(M)/\!\!/\mathrm{Diff}(M)$ rather than a coarse space is folklore in the mathematical-physics community and is made explicit in, for example, the nLab treatment of the moduli space of metrics and in Berktav's construction [20] of the moduli stack of vacuum Einstein solutions in n dimensions (with a sharp stacky equivalence to a gauge theory in three dimensions). The deformation-theoretic side — a graded Lie algebra whose Maurer–Cartan equation is the vacuum Einstein equation, with linearized gravity as its first cohomology — is developed by Reiterer and Trubowitz [19] and, in L_∞ language, by Jurčo, Raspollini, Sämann and Wolf [23]. The Hamiltonian/constraint side, where the diffeomorphism symmetry of the evolution equations is genuinely a groupoid (or Lie algebroid) and *not* the momentum map of any group action, is the subject of Blohmann, Fernandes and Weinstein [17] and Blohmann and Weinstein [18]. Our contribution is not a new theorem in any of these directions; it is a consolidation, with explicit epistemic labels and explicit isotropy computations, of why the stack — and specifically its automorphism data — is the object the reconstruction program needs, and a statement of the precise sense in which the classical regime composes with the others.

2 Mathematical framework

We fix a smooth connected oriented manifold M of dimension $n = 4$ unless stated otherwise; the constructions are dimension-agnostic except where a specific example fixes n.

2.1 Lorentzian metrics and causal structure

Definition 2.1 (Lorentzian metric)**.** A *Lorentzian metric* on M is a smooth symmetric $(0,2)$-tensor field g that is nondegenerate and has signature $(-,+,+,+)$ at every point. The pair (M,g) is a *Lorentzian manifold*. We write $\mathrm{Met_L}(M) \subset \Gamma(\mathrm{Sym}^2\, T^*M)$ for the set of all Lorentzian metrics on M; it is an open subset (in a suitable topology) of the space of smooth symmetric $(0,2)$-tensors, and in particular an infinite-dimensional smooth manifold modelled on $\Gamma(\mathrm{Sym}^2\, T^*M)$.

At a point p the metric partitions $T_pM \setminus \{0\}$ into vectors that are *timelike* ($g(v,v) < 0$), *null* ($g(v,v) = 0$), and *spacelike* ($g(v,v) > 0$). A *time orientation* is a continuous choice of one of the two components of the timelike cone at each point; we always assume (M,g) is time oriented. A piecewise-smooth curve is *causal* if its tangent is everywhere timelike or null, and *future directed* if it lies in the chosen cone. The causal relation $p \preceq q$ (there is a future-directed causal curve from p to q, or $p = q$) is the primitive causal datum; the light-cone (conformal) structure of g determines $\preceq$ and conversely $\preceq$ determines g up to a pointwise positive conformal factor.

Definition 2.2 (Causality conditions)**.** A spacetime (M,g) is *chronological* if it has no closed timelike curve, *causal* if it has no closed causal curve, *strongly causal* at p if every neighbourhood of p contains a neighbourhood no causal curve enters more than once, and *globally hyperbolic* if it is causal and every causal diamond $J^+(p) \cap J^-(q)$ is compact, where $J^\pm$ denote causal future/past.

Definition 2.3 (Cauchy surface)**.** A subset $\Sigma \subset M$ is a *Cauchy surface* if it is met exactly once by every inextendible causal curve. A spacetime that admits a Cauchy surface is the canonical arena for the initial-value formulation.

The following two facts anchor the well-posedness of the classical theory and are used repeatedly; both are standard.

Theorem 2.4 (Bernal–Sánchez smooth splitting)**.** **[S]** *If (M,g) is globally hyperbolic then it admits a smooth spacelike Cauchy hypersurface Σ, and there is a diffeomorphism $M \cong \mathbb{R} \times \Sigma$ under which each $\{t\} \times \Sigma$ is a smooth spacelike Cauchy surface and the metric takes the form $g = -N^2\, dt^2 + h_t$ with $N > 0$ a smooth lapse and h_t a smooth t-family of Riemannian metrics on Σ.*

Reference. Geroch [5] proved the topological equivalence of global hyperbolicity with the existence of a Cauchy surface and produced a continuous Cauchy time function; Bernal and Sánchez [6] upgraded the time function and the level surfaces to the smooth spacelike category, giving the stated smooth product splitting. See [1, Ch. 8] and [2] for the causal-theory background. □

2.2 The Einstein equation

Definition 2.5 (Einstein tensor and equation)**.** For $g \in \mathrm{Met_L}(M)$ let Ric_g denote the Ricci curvature and $R_g = g^{ab}(\mathrm{Ric}_g)_{ab}$ the scalar curvature. The *Einstein tensor* is

$$\mathrm{Ein}[g] \;:=\; \mathrm{Ric}_g - \tfrac{1}{2} R_g\, g.$$

Given a stress-energy tensor T (a symmetric $(0,2)$-tensor built from matter fields, possibly $T \equiv 0$ in vacuum) and Newton's constant G, the *Einstein equation* is the system of quasilinear second-order partial differential equations

$$\mathrm{Ein}[g] \;=\; 8\pi G\, T. \tag{1}$$

The contracted Bianchi identity $\nabla^a(\mathrm{Ein}[g])_{ab} = 0$ forces $\nabla^a T_{ab} = 0$ on solutions, so (1) is not independent across its components; four of the ten equations constrain initial data rather than evolve it.

Definition 2.6 (Solution locus)**.** With matter content fixed (either $T \equiv 0$, the vacuum case, or T a prescribed functional of auxiliary matter fields), the *solution locus* is

$$\mathrm{Sol}(\mathrm{Ein}) \;:=\; \{\, g \in \mathrm{Met}_{\mathrm{L}}(M) : \mathrm{Ein}[g] = 8\pi G\, T \,\} \subset \mathrm{Met}_{\mathrm{L}}(M).$$

In the vacuum case with vanishing cosmological constant this is the Ricci-flat locus $\{\mathrm{Ric}_g = 0\}$.

The initial-value formulation splits (1) into constraints on a slice and evolution off it. On a spacelike hypersurface Σ with induced metric h and second fundamental form K, the Gauss–Codazzi relations turn the 00 and $0a$ components of (1) into the *Hamiltonian* and *momentum* constraints

$$R_h + (\mathrm{tr}_h K)^2 - |K|_h^2 = 16\pi G\,\rho, \qquad D^b(K_{ab} - h_{ab}\,\mathrm{tr}_h K) = 8\pi G\, j_a, \tag{2}$$

where D denotes the Levi-Civita connection of the spatial metric h (distinct from the spacetime connection ∇ used above), R_h is the intrinsic scalar curvature of h, and ρ, j are the matter energy and momentum densities. Constraint-satisfying data (Σ, h, K) is the input to the Cauchy problem.

Theorem 2.7 (Choquet-Bruhat–Geroch maximal development)**.** **[S]** *Let (Σ, h, K) be smooth vacuum initial data satisfying the constraints (2) with $T \equiv 0$. Then there is a maximal globally hyperbolic vacuum development (M, g) of the data, unique up to isometry, into which every other globally hyperbolic vacuum development of the same data embeds.*

Reference. Local existence and uniqueness of a globally hyperbolic development is Choquet-Bruhat [3] (harmonic-gauge reduction to a quasilinear hyperbolic system); the global maximal development is Choquet-Bruhat and Geroch [4]. The 1969 argument uses Zorn's lemma; a choice-free construction was later given by Sbierski [7]. "Unique up to isometry" is the seed of the entire stack story: the theorem does not produce a unique metric, it produces a unique *isometry class*, i.e. a unique object of the solution groupoid. □

Theorem 2.7 already tells us what the correct output type of the Cauchy problem is. It is not a point of a set of metrics. It is an object of a groupoid, defined only up to the isomorphisms of that groupoid, which are exactly diffeomorphisms. The rest of the paper takes that observation seriously.

2.3 The configuration groupoid

Definition 2.8 (Diffeomorphism action)**.** The group $\mathrm{Diff}(M)$ of smooth diffeomorphisms of M acts on $\mathrm{Met}_{\mathrm{L}}(M)$ on the right by pullback,

$$\mathrm{Met}_{\mathrm{L}}(M) \times \mathrm{Diff}(M) \longrightarrow \mathrm{Met}_{\mathrm{L}}(M), \qquad (g, \varphi) \longmapsto \varphi^* g.$$

The action preserves Sol(Ein): if $\mathrm{Ein}[g] = 8\pi G T$ then $\mathrm{Ein}[\varphi^* g] = 8\pi G\, \varphi^* T$, and for diffeomorphism-covariant matter $\varphi^* T$ is the stress tensor of the pulled-back matter fields, so $\varphi^* g$ solves the equation with the transported matter content.

Definition 2.9 (Isometry group)**.** The *isometry group* of (M, g) is the stabilizer of g under this action,

$$\mathrm{Isom}(M, g) \;:=\; \mathrm{Stab}_{\mathrm{Diff}(M)}(g) \;=\; \{\varphi \in \mathrm{Diff}(M) : \varphi^* g = g\}.$$

By the Myers–Steenrod theorem $\mathrm{Isom}(M, g)$ is a Lie group, and its Lie algebra is the space of *Killing vector fields*

$$\mathrm{Kill}(M, g) \;=\; \{\xi \in \Gamma(TM) : \mathcal{L}_\xi g = 0\},$$

where $\mathcal{L}_\xi$ is the Lie derivative; the Killing condition is the first-order linear system $\nabla_a \xi_b + \nabla_b \xi_a = 0$.

The physical reading of theorem 2.9 is the one the reconstruction program cares about. A Killing field is not a gauge redundancy. It is a *physical residual symmetry*: it generates a one-parameter family of isometries under which the solution is genuinely invariant, it produces a conserved Noether current for the matter it acts on, and, at the level of the gravitational gauge algebra, it makes that algebra reducible (there is a gauge transformation, generated by ξ, that acts trivially on g). Collapsing g to its orbit forgets $\mathrm{Isom}(M, g)$; keeping the groupoid remembers it as the automorphism group of the object.

3 The solution stack $\mathrm{Sol(Ein)} /\!\!/ \mathrm{Diff}(M)$

3.1 Action groupoids and quotient stacks

We recall the finite-dimensional template so that the gravitational statement is a specialization rather than a definition made up for the occasion.

Definition 3.1 (Action groupoid)**.** Let a group G act on a set (or space) X on the right. The *action groupoid* $X /\!\!/ G$ has object set X and morphism set $X \times G$, with the morphism (x, γ) going $x\gamma \to x$ (source $s(x, \gamma) = x\gamma$, target $t(x, \gamma) = x$), identities (x, e), and composition $(x, \gamma) \circ (x\gamma, \delta) = (x, \gamma\delta)$. Its isomorphism classes of objects are the G-orbits, and its automorphism group at x is the stabilizer:

$$\mathrm{Aut}_{X /\!\!/ G}(x) \;=\; \{\gamma \in G : x\gamma = x\} \;=\; \mathrm{Stab}_G(x). \tag{3}$$

The associated *quotient stack* $[X/G]$ is the stackification of this groupoid for the relevant topology; for our purposes the essential data is the groupoid, and (3) is the fact we use.

The contrast with the coarse quotient is exactly (3). The coarse quotient set X/G remembers only the orbit; it has forgotten $\mathrm{Stab}_G(x)$. Two actions with the same orbits but different stabilizers give the same coarse quotient and different groupoids. In the presence of nontrivial, jumping stabilizers, the coarse quotient is the wrong invariant precisely because (3) is nontrivial.

Definition 3.2 (The solution stack)**.** The *solution stack* of general relativity (with fixed matter content) is the action groupoid

$$\mathrm{Sol(Ein)}/\!\!/\mathrm{Diff}(M): \qquad \mathrm{Ob} = \mathrm{Sol(Ein)}, \quad \mathrm{Mor} = \{(g,\varphi) : g \in \mathrm{Sol(Ein)},\ \varphi \in \mathrm{Diff}(M)\},$$

with $(g,\varphi)\colon \varphi^*g \to g$. Its isomorphism classes are the physical (diffeomorphism-invariant) gravitational configurations; its automorphism groups are the isometry groups.

Proposition 3.3 (Isotropy of the solution stack)**. [S]** *For every* $g \in \mathrm{Sol(Ein)}$*,*

$$\mathrm{Aut}_{\mathrm{Sol(Ein)}/\!\!/\mathrm{Diff}(M)}(g) \;\cong\; \mathrm{Isom}(M,g),$$

with Lie algebra the Killing algebra $\mathrm{Kill}(M,g)$*. In particular the stack is a* point with automorphisms*: the physical content of a solution is its isometry class* together with *its isometry group, not the isometry class alone.*

Proof. Specialize (3) to $X = \mathrm{Sol(Ein)}$, $G = \mathrm{Diff}(M)$, and $x = g$. The automorphism group at g is $\mathrm{Stab}_{\mathrm{Diff}(M)}(g) = \{\varphi : \varphi^*g = g\}$, which is $\mathrm{Isom}(M,g)$ by theorem 2.9. The identification of the Lie algebra with Killing fields is the infinitesimal form: differentiating $\varphi_s^*g = g$ along a one-parameter family φ_s with generator ξ gives $\mathcal{L}_\xi g = 0$. □

Theorem 3.3 is elementary once the objects are set up correctly, and that is the point: the content is entirely in choosing the groupoid over the set. We now show that this choice is forced by the local geometry of the solution space, not merely convenient.

3.2 The slice theorem and why the coarse quotient tears

Theorem 3.4 (Ebin slice theorem, isotropy form)**. [S]** *Let M be a closed manifold and let* $\mathrm{Met}(M)$ *be the space of* Riemannian *metrics on M. At each $g \in \mathrm{Met}(M)$ the action of* $\mathrm{Diff}(M)$ *on the space of metrics admits a slice $\mathcal{S}_g$: a submanifold through g, invariant under the compact isotropy group* $\mathrm{Isom}(M,g)$*, such that the map* $\mathcal{S}_g \times_{\mathrm{Isom}(M,g)} \mathrm{Diff}(M) \to \mathrm{Met}(M)$ *is a diffeomorphism onto a* $\mathrm{Diff}(M)$*-invariant neighbourhood of the orbit. Consequently the orbit space is locally modelled near $[g]$ on* $\mathcal{S}_g/\mathrm{Isom}(M,g)$*, and the isotropy group of the action at g is* $\mathrm{Isom}(M,g)$*.*

Reference. Ebin [8]; the underlying L^2/Sobolev manifold structure on $\mathrm{Met}(M)$ and on the orbits is Fischer and Marsden [9]. The Riemannian statement uses properness of the $\mathrm{Diff}(M)$-action, which holds on a closed manifold. The Lorentzian and non-compact cases are subtler (see section 9); the isotropy identification $\mathrm{Aut}([g]) \cong \mathrm{Isom}(M,g)$ survives as a groupoid statement regardless. □

Theorem 3.5 (Linearization stability; conical structure of the solution space)**. [S]** *Let (M,g) be a vacuum solution with a compact Cauchy surface of constant mean curvature. Then:*

(i) *the linearized Einstein operator at g has a solution extending a given infinitesimal deformation to second order if and only if the deformation is L^2-orthogonal to the space spanned by the Killing fields' associated conserved quantities — the Taub conserved quantities, quadratic functionals of the linearized field built from each Killing field;*

(ii) *(M, g) is linearization stable (every solution of the linearized equation is tangent to a curve of exact solutions) if and only if* $\mathrm{Kill}(M, g) = 0$*;*

(iii) *at a solution with* $\dim \mathrm{Kill}(M, g) = k > 0$ *the solution space has a quadratic singularity: near g,* $\mathrm{Sol}(\mathrm{Ein})$ *is modelled on the zero set of a k-vector-valued quadratic form (the second-order obstruction, the Kuranishi map), so the coarse quotient* $\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)$ *has a conical singularity there.*

Reference. Moncrief [10, 11] identified the second-order obstruction with the Killing fields and proved (ii); Fischer, Marsden and Moncrief [12] and Arms, Marsden and Moncrief [13] established the conical (quadratic) local structure (iii) and its symplectic stratification. The CMC and compactness hypotheses are essential to these theorems as stated; the asymptotically flat case is analogous in spirit but governed by boundary terms and ADM charges rather than by these exact statements. □

The two theorems fit together into a single picture, which we state as the paper's organizing proposition.

Proposition 3.6 (The stack resolves the coarse singular locus)**. [S/H]** *Under the hypotheses of theorem 3.5, the singular locus of the coarse orbit space* $\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)$ *is exactly the symmetric locus* $\{[g] : \mathrm{Kill}(M, g) \neq 0\}$*. On that locus the solution groupoid* $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$ *carries the nontrivial automorphism group* $\mathrm{Isom}(M, g)$ *of theorem 3.3, and the quadratic obstruction of theorem 3.5 (iii) is the deformation-theoretic manifestation of that automorphism group. The stack is therefore the object on which the singular points of the coarse quotient are recorded rather than lost: the data the coarse quotient tears at is precisely the automorphism data the groupoid keeps.*

Discussion. The identification of the singular locus with the symmetric locus is theorem 3.5 (iii): away from Killing symmetry the space is a smooth manifold, at Killing symmetry it is a quadratic cone. By theorem 3.3 the automorphism group of the groupoid jumps up exactly on that locus, from the trivial group (or a discrete group) to a positive-dimensional Lie group. The Kuranishi quadratic form of theorem 3.5 (iii) is valued in $H^0 = \mathrm{Kill}(M, g) = \mathrm{LieAut}([g])$; it is nonzero precisely because that cohomology is nonzero. Hence the "defect" that makes the coarse quotient singular and the "automorphism data" that the groupoid records are two descriptions of the same H^0. The label is **[S/H]** rather than **[S]** because the clean global statement requires the compact-CMC hypotheses of theorem 3.5; the underlying identification of automorphisms with isometries (theorem 3.3) is unconditionally **[S]**. □

3.3 A diagrammatic summary

The structure is compactly summarized by the two maps out of the morphism space of the groupoid and the failure of the coarse quotient to be their coequalizer in a way that remembers isotropy.

$$
\begin{tikzcd}
\mathrm{Sol}(\mathrm{Ein}) \times \mathrm{Diff}(M) \arrow[r, shift left, "t"] \arrow[r, shift right, "s"'] & \mathrm{Sol}(\mathrm{Ein}) \arrow[r, "\pi"] \arrow[d, dashed] & \big[\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)\big] \arrow[d, "\text{forget Aut}"] \\
& {} & \mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)
\end{tikzcd}
$$

Here $s(g,\varphi) = \varphi^* g$ and $t(g,\varphi) = g$ are source and target, the top row is the groupoid, the stack $[\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)]$ is its stackification, and the vertical map to the coarse quotient is the functor that forgets the automorphism groups. Theorem 3.6 says the vertical map is an isomorphism away from the symmetric locus and genuinely loses information (the isotropy groups, and with them the smoothness of the total object) on it.

4 Worked examples: isotropy and stratification

We compute $\mathrm{Isom}(M,g)$ for the standard solution families. All statements in this section are **[S]**; they are classical, and their point here is to make the automorphism data of theorem 3.3 concrete and to exhibit an explicit isotropy jump.

Remark 4.1 (Compact-CMC versus asymptotically flat)*.* The standard families below (Minkowski, Schwarzschild, Kerr) are asymptotically flat and do not have compact Cauchy surfaces, so the exact linearization stability statements of theorem 3.5, proved under compact constant-mean-curvature hypotheses, do not apply to them verbatim. What survives unconditionally is the isotropy identification of theorem 3.3: the automorphism group of the stack object is $\mathrm{Isom}(M,g)$ regardless of asymptotics. The stratification principle — higher-symmetry solutions sitting in the closures of lower-symmetry strata — also carries over to the asymptotically flat setting, but there it is organized by ADM charges and boundary conditions at infinity rather than by the compact-slice obstruction theory. We use the examples to illustrate the automorphism and stratification structure, not to invoke theorem 3.5 outside its hypotheses.

Example 4.2 (Minkowski space)**.** For $(M,g) = (\mathbb{R}^4,\eta)$ with $\eta = \mathrm{diag}(-1,1,1,1)$, the Killing equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$ has the maximal solution space in $n = 4$, of dimension $\frac{n(n+1)}{2} = 10$: four translations ∂_μ and six Lorentz generators $x_\mu \partial_\nu - x_\nu \partial_\mu$, where the coordinate indices are lowered with the Minkowski metric, $x_\mu = \eta_{\mu\rho} x^\rho$. Hence

$$
\mathrm{Isom}(\mathbb{R}^4,\eta) \;=\; \mathbb{R}^4 \rtimes O(3,1), \qquad \mathrm{Isom}_0(\mathbb{R}^4,\eta) \;=\; \mathbb{R}^4 \rtimes SO(3,1)^{\uparrow},
$$

the Poincaré group and its connected (proper orthochronous) component; $\mathrm{Aut}([\eta])$ is Poincaré. Minkowski space is the maximally symmetric flat solution; in the stack it is a point with a 10-dimensional automorphism group.

Example 4.3 (Schwarzschild)**.** The Schwarzschild metric (exterior, mass $m > 0$)

$$
g = -\Big(1 - \tfrac{2Gm}{r}\Big) dt^2 + \Big(1 - \tfrac{2Gm}{r}\Big)^{-1} dr^2 + r^2\, d\Omega^2
$$

is static and spherically symmetric. Its connected isometry group is

$$\mathrm{Isom}_0 = \mathbb{R} \times SO(3), \qquad \dim = 1 + 3 = 4,$$

generated by the timelike Killing field ∂_t and the three rotation Killing fields of S^2. The full isometry group includes a discrete time-reflection. By theorem 2.7 the object of the solution stack is really the maximal globally hyperbolic development, here the Kruskal–Szekeres extension; the connected isometry group is unchanged, $\mathbb{R} \times SO(3)$, because ∂_t extends across the horizon to a global Killing field (spacelike inside the horizon, the boost Killing field of the Kruskal plane), so the automorphism data of the maximal object matches that of the exterior patch. In the stack, each Schwarzschild mass m is a point with automorphism group $\mathbb{R} \times SO(3)$, and the masses form a one-parameter family of such points.

Example 4.4 (Kerr)**.** The Kerr metric (mass m, angular momentum per unit mass a, in Boyer–Lindquist coordinates) is stationary and axisymmetric but not static and not spherically symmetric for $a \neq 0$. Its connected isometry group is

$$\mathrm{Isom}_0 = \mathbb{R} \times U(1), \qquad \dim = 1 + 1 = 2,$$

generated by ∂_t (stationarity) and ∂_ϕ (axisymmetry). The two Killing fields commute, so the group is abelian.

Example 4.5 (FLRW cosmologies)**.** For a Friedmann–Lemaître–Robertson–Walker metric $g = -dt^2 + a(t)^2\, h_k$, where h_k is the maximally symmetric spatial metric of constant curvature $k \in \{+1, 0, -1\}$, the spatial slices carry a six-dimensional isometry group:

$$\mathrm{Isom(slice)} = \begin{cases} SO(4), & k = +1, \\ \mathbb{R}^3 \rtimes SO(3) = E(3), & k = 0, \\ SO(3,1)^{\uparrow}, & k = -1, \end{cases} \qquad \dim = \tfrac{3\cdot 4}{2} = 6.$$

For generic $a(t)$ there is no additional timelike Killing field (the geometry is not stationary), so $\mathrm{Isom}(M, g)$ is the six-dimensional spatial group acting at each t; for special $a(t)$ (de Sitter, $a(t) = e^{Ht}$ with $k = 0$) the isometry group enhances to the ten-dimensional de Sitter group $SO(4,1)$. This enhancement is itself an isotropy jump, of the same kind as in theorem 4.6 below. Note that de Sitter spacetime admits FLRW foliations for all three spatial curvatures $k \in \{+1, 0, -1\}$: these are different slicings of one and the same maximally symmetric spacetime, all sharing the single $SO(4,1)$ isometry group. The foliation is a coordinate (gauge) choice; the automorphism group of the stack object is $SO(4,1)$ regardless of which k-slicing one writes down.

Example 4.6 (The Kerr-to-Schwarzschild isotropy jump)**.** Consider the two-parameter Kerr family $g_{m,a}$ with $m > 0$ fixed and a ranging over a neighbourhood of 0. For $a \neq 0$ we have $\mathrm{Isom}_0(g_{m,a}) = \mathbb{R} \times U(1)$, of dimension 2; at $a = 0$ the metric is Schwarzschild and $\mathrm{Isom}_0(g_{m,0}) = \mathbb{R} \times SO(3)$, of dimension 4. The isometry algebra jumps:

$$\dim \mathrm{Kill}(g_{m,a}) = \begin{cases} 2, & a \neq 0, \\ 4, & a = 0. \end{cases}$$

Solution	Connected Isom_0	dim Kill	Type
Minkowski	$\mathbb{R}^4 \rtimes SO(3,1)^{\uparrow}$	10	maximally symmetric (flat)
de Sitter	$SO(4,1)$	10	maximally symmetric ($\Lambda > 0$)
FLRW (generic a)	spatial 6-dim group	6	homogeneous isotropic
Schwarzschild	$\mathbb{R} \times SO(3)$	4	static spherical
Kerr ($a \neq 0$)	$\mathbb{R} \times U(1)$	2	stationary axisymmetric
generic vacuum	$\{e\}$	0	no symmetry

Table 1: Isotropy (automorphism) data of representative points of the solution stack. The stratification is by dim Kill; higher-symmetry solutions sit in the closures of lower-symmetry strata (theorem 4.6). All entries are **[S]**.

The two extra Killing fields at $a = 0$ are the two rotation generators L_x, L_y of S^2 orthogonal to the axis; $\partial_\phi = L_z$ persists for all a, while L_x, L_y exist only in the round limit. In the coarse quotient Sol(Ein)/Diff, the point $[g_{m,0}]$ is a more singular stratum than the nearby $[g_{m,a}]$: the isotropy jumps up there, and by theorem 3.6 it is exactly at such a point that the coarse quotient fails to be a manifold. In the solution stack the jump is recorded as a jump in the automorphism group, $\mathbb{R} \times U(1) \rightsquigarrow \mathbb{R} \times SO(3)$, and the stack is stratified by isometry type, with $[g_{m,0}]$ lying on a lower-dimensional, higher-symmetry stratum in the closure of the generic Kerr stratum.

Theorem 4.6 is the concrete form of the reconstruction program's requirement. Any deeper theory — a spin-foam amplitude, a holographic code, a renormalization-group endpoint — that claims Schwarzschild as a semiclassical limit must reproduce not just the isometry class of Schwarzschild but its $\mathbb{R} \times SO(3)$ automorphism group, and must reproduce the fact that this group is strictly larger than the automorphism group of nearby rotating solutions. A theory that produced the right coarse geometry but the wrong isotropy would be reproducing the coarse quotient, not the stack, and would be missing exactly the data — degeneracies, conserved charges, gauge-algebra reducibility — that has physical consequences.

5 The deformation complex and the tangent stack

The linearization-stability theorem (theorem 3.5) is best understood as a statement about a cochain complex — the tangent complex of the solution stack at g — whose cohomology organizes Killing fields, on-shell perturbations, and obstructions into a single object. This is the deformation-theoretic content of the stack, and it is the level at which the classical regime connects to the Batalin–Vilkovisky and L_∞ machinery of the neighbouring regime.

5.1 The linearized theory as a two-step complex

Fix $g \in$ Sol(Ein) vacuum. Write DE_g for the linearized Einstein operator and DE_g^* for its formal adjoint, and let $\delta_g^* : \Gamma(TM) \to \Gamma(\mathrm{Sym}^2 T^*M)$, $\xi \mapsto \mathcal{L}_\xi g$, be the map sending a vector field to the metric variation it generates (the linearized gauge action), and let $\delta_g : \Gamma(\mathrm{Sym}^2 T^*M) \to \Gamma(TM)$ be the (metric) divergence operator, the L^2 formal adjoint of

δ_g^*. Since $(\delta_g^*\xi)_{ab} = \nabla_a\xi_b + \nabla_b\xi_a$, its exact adjoint is $h_{ab} \mapsto -2\nabla^a h_{ab}$; we absorb the constant -2 into δ_g, as it does not affect any cohomology below. The relevant complex is

$$\Gamma(TM) \xrightarrow{\delta_g^*} \Gamma(\mathrm{Sym}^2 T^*M) \xrightarrow{DE_g} \Gamma(\mathrm{Sym}^2 T^*M), \qquad DE_g \circ \delta_g^* = 0, \tag{4}$$

where $DE_g \circ \delta_g^* = 0$ is the linearized *gauge invariance* of the Einstein operator: it expresses that an infinitesimal diffeomorphism is annihilated by the linearized equation. Its Noether-dual companion is the linearized *contracted Bianchi identity* $\delta_g \circ DE_g = 0$, a separate statement about the divergence of the Einstein operator; the two identities are adjoint and should not be conflated. Together they extend (4) to the four-term elliptic (Batalin–Vilkovisky field–antifield) resolution

$$\Gamma(TM) \xrightarrow{\delta_g^*} \Gamma(\mathrm{Sym}^2 T^*M) \xrightarrow{DE_g} \Gamma(\mathrm{Sym}^2 T^*M) \xrightarrow{\delta_g} \Gamma(TM), \tag{5}$$

in degrees $0, 1, 2, 3$, whose degree-0 chain space carries the ghosts (gauge parameters), degree-1 the metric perturbations, degree-2 the antifields (equations of motion), and degree-3 the antighosts $\Gamma(TM)$, whose cohomology is $\operatorname{coker} \delta_g$; this is the linearized shadow of the BV complex of theorem 5.3. Here "elliptic" refers to the formal principal symbol of the complex (equivalently, to its Riemannian-signature counterpart), under which (5) is an elliptic complex; in Lorentzian signature the vacuum Einstein operator is hyperbolic — weakly hyperbolic before gauge fixing, strictly hyperbolic in harmonic gauge — and the physical evolution is governed by that hyperbolicity (theorem 2.7), not by ellipticity. The symbolic/elliptic reading is what makes the cohomology count below well posed; the analytic evolution is a separate, hyperbolic, matter (section 9). For the moduli count we read the truncation (4) as a cochain complex in degrees $0, 1, 2$, which gives the following cohomology.

Proposition 5.1 (Cohomology of the linearized complex)**. [S/H]** *For the complex* (4)*:*

$$\begin{aligned}
H^0 &= \ker \delta_g^* = \mathrm{Kill}(M, g) = \mathrm{LieAut}_{\mathrm{Sol}/\!\!/\mathrm{Diff}}([g]),\\
H^1 &= \ker DE_g / \operatorname{im} \delta_g^* = \{\textit{on-shell linear perturbations modulo gauge}\},\\
H^2 &= \operatorname{coker} DE_g \supseteq \ker \delta_g / \operatorname{im} DE_g \supseteq \{\textit{second-order obstructions}\},
\end{aligned}$$

so H^0 is the isotropy Lie algebra, H^1 is the (formal) tangent space to the moduli stack at $[g]$, and the second-order (Kuranishi) map $\kappa : H^1 \to \ker \delta_g / \operatorname{im} DE_g \subseteq H^2$ is the quadratic obstruction whose vanishing controls whether a first-order deformation integrates; the target is the middle cohomology of the elliptic resolution (5)*, which sits inside* $\operatorname{coker} DE_g$ *by the linearized Bianchi identity. By theorem 3.5, in the compact-CMC vacuum case κ is nonzero precisely when $H^0 = \mathrm{Kill}(M, g) \neq 0$, and its image pairs with H^0; this is the deformation-theoretic form of theorem 3.6.*

Discussion. $H^0 = \ker \delta_g^*$ is by definition the vector fields ξ with $\mathcal{L}_\xi g = 0$, the Killing fields; by theorem 3.3 this is $\mathrm{LieAut}([g])$. The identification of H^1 with linear solutions modulo linear gauge is the standard linearized moduli count, and H^2 receives the second-order term of the expansion of Ein in the deformation; in the four-term elliptic resolution (5) this obstruction space is the middle cohomology $\ker \delta_g / \operatorname{im} DE_g$, finite-dimensional by ellipticity and paired with H^0 by the adjointness of δ_g and δ_g^*. That κ pairs with H^0 and vanishes iff $H^0 = 0$ (in the compact-CMC case) is exactly theorem 3.5 (i)–(iii): the Killing fields obstruct integrability at second order. The label is **[S/H]**: the algebra of (4) is **[S]**, but the identification of H^2 with the full obstruction space and the pairing with H^0 use the analytic input of theorem 3.5, which carries hypotheses. □

5.2 Maurer–Cartan and the derived tangent

Theorem 5.1 is the shadow of a richer statement: the vacuum Einstein equation is the Maurer–Cartan equation of a differential graded (or L_∞) algebra, and $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$ is its (formal) derived moduli.

Proposition 5.2 (Graded-Lie/L_∞ presentation of the vacuum moduli)**.** **[S/H]** *There is a graded Lie algebra* $(\mathfrak{L}, d, [\cdot,\cdot])$*, natural in the conformal-frame-bundle data of* M*, whose Maurer–Cartan elements* $a \in \mathfrak{L}^1$ *with* $da + \frac{1}{2}[a,a] = 0$ *are in bijection with vacuum Einstein metrics near a background, whose gauge transformations are the exponentiated action of* $\mathfrak{L}^0$*, and for which the cohomology of the twisted differential* $d_a = d + [a,\cdot]$ *at a solution* a *reproduces theorem 5.1:* $H^0(d_a)$ *is the Killing algebra and* $H^1(d_a)$ *is linearized gravity modulo gauge. Consequently the formal neighbourhood of* $[g]$ *in* $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$ *is the derived Maurer–Cartan (Kuranishi) space of* $\mathfrak{L}$ *at* a*.*

Reference. This is the content of Reiterer and Trubowitz [19], who construct such a graded Lie algebra with the vacuum Einstein equation as its Maurer–Cartan equation and identify the first homology at a Maurer–Cartan element with linearized gravity; the L_∞/Batalin–Vilkovisky packaging of classical field theories in these terms is Jurčo, Raspollini, Sämann and Wolf [23]. The identification of the formal moduli functor of a differential graded Lie or L_∞ algebra with a derived deformation problem is the standard deformation-theory paradigm (Hinich; Pridham; Lurie); we invoke it at the formal level only, which is why the label is **[S/H]** and not **[S]**: the global, non-formal, analytic construction is the open "homological PDE" program of [19], not a finished theorem. □

Remark 5.3 (Where this meets the next regime)*.* Theorem 5.2 is the exact point of contact between the classical geometry regime of this paper and the gauge/observables regime built on the Batalin–Vilkovisky–BRST formalism and shifted symplectic geometry. The derived critical locus of the Einstein–Hilbert action, in the sense of Pantev, Toën, Vaquié and Vezzosi [22], carries a (-1)-shifted symplectic structure and is the derived enhancement of $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$; its ghost-for-ghost tower is nontrivial exactly in the degrees dual to $\mathrm{Kill}(M,g) = H^0$. The isotropy data that theorem 3.3 records as an automorphism group and theorem 5.1 records as H^0 is the same data that the neighbouring regime records as a reducibility of the gravitational gauge algebra. Three descriptions, one invariant.

5.3 Hamiltonian form: the constraint groupoid

There is a second, Hamiltonian, route to the same conclusion that the symmetry object of gravity is a groupoid rather than a group, and it is worth recording because it is where the diffeomorphism symmetry differs sharply from ordinary gauge symmetry.

Proposition 5.4 (Constraints as a Lie-algebroid, not a momentum map). **[S/H]** *In the ADM formulation, on the phase space $T^*\mathrm{Met}(\Sigma)$ of a Cauchy slice with canonical data (h_{ab}, π^{ab}), the Hamiltonian and momentum constraints generate the "hypersurface deformation" bracket algebra. This bracket algebra is* not *the Lie algebra of a group action — its structure functions depend on the phase-space point h — but it is the bracket of a Lie algebroid over $\mathrm{Met}(\Sigma)$, the algebroid of a groupoid of diffeomorphisms between spacelike hypersurfaces. The Einstein constraints are therefore the sections of a Lie algebroid rather than the components of a momentum map.*

Reference. Blohmann, Fernandes and Weinstein [17] show that the hypersurface deformation brackets reproduce the bracket relations of the Lie algebroid of a groupoid of embeddings of spacelike hypersurfaces, and that, unlike a Yang–Mills constraint set, the gravitational constraints are not the momentum map of any Hamiltonian group action; the systematic "Hamiltonian Lie algebroid" framework is developed by Blohmann and Weinstein [18]. The physical reading: even at the level of the constraint algebra, the correct symmetry object of gravity is a groupoid/algebroid, consistent with the covariant-configuration statement of theorem 3.2. The label **[S/H]** reflects that this is an established mathematical structure whose use as the organizing principle of a quantization is a physical proposal, not a theorem. □

6 The connection reformulation and the bridge to quantum geometry

The classical regime does not connect to the quantum-geometry regimes through the metric directly; it connects through a first-order, gauge-theoretic reformulation. We record it because it is the technical door from this paper's stack into the spin-network phase space, and because it re-expresses the same isotropy invariant in gauge-theoretic language.

Proposition 6.1 (First-order / Palatini–Cartan reformulation). **[S/H]** *On the sector where the coframe is nondegenerate, general relativity is equivalent to a first-order theory of an $SO(3,1)$ (or $\mathrm{Spin}(3,1)$) connection ω and a soldering coframe e, with action*

$$S_{\mathrm{PC}}[e,\omega] = \int_M \epsilon_{IJKL}\, e^I \wedge e^J \wedge F^{KL}(\omega), \qquad F(\omega) = d\omega + \omega \wedge \omega.$$

Varying ω imposes vanishing torsion, which solves ω in terms of e; varying e then reproduces the vacuum Einstein equation $\mathrm{Ein}[g] = 0$ for $g = \eta_{IJ} e^I \otimes e^J$. Passing to the Hamiltonian form on a Cauchy slice and performing the Ashtekar–Barbero canonical transformation yields an $SU(2)$ connection A^i_a and a densitized triad E^a_i, with a real Barbero parameter γ, on which the spin-network kinematics of the neighbouring regime is built.

Reference. The equivalence of first-order Palatini–Cartan gravity with the metric theory on the nondegenerate sector is standard; see Krasnov [21] for a systematic modern account of the connection formulations. The self-dual complex connection variables are Ashtekar [15]; the real Lorentzian $SU(2)$ variables with the Barbero parameter are Barbero [16]. The **[S/H]** label reflects that the reformulation is exact on the nondegenerate sector but that the extension across degenerate coframes (where e drops rank) is formulation-dependent and is one of the places the metric and connection pictures genuinely differ. □

Remark 6.2 (Isotropy in the connection picture). Under theorem 6.1 a Killing field ξ of g lifts to a symmetry of the pair (e, ω) combining a diffeomorphism with a compensating local $SO(3,1)$ frame rotation. The automorphism group of theorem 3.3 thus reappears in the connection description as a stabilizer inside the semidirect product of $\mathrm{Diff}(M)$ with the gauge group of frame rotations. The invariant is the same; only its packaging changes. This is the sense in which the stack, not the metric, is what is transported across the bridge to quantum geometry.

7 Modular composition and the weakest-link status calculus

This regime is one of twelve in a modular program. The program is deliberately *not* a unification: it does not assert a single master theory of which each regime is a special case. Instead it tracks how epistemic warrant propagates when regimes are chained, using a small, explicit calculus that we now state and apply.

7.1 The status calculus

Definition 7.1 (Warrants and status)**.** Let the ordered set of *warrants* be $\mathsf{S} < \mathsf{H} < \mathsf{P}$, read respectively as "standard mathematics / established mathematical physics", "strong physical heuristic", and "speculative ontological extension". A *status* is a nonempty set of warrants, written by its range: S, H, P, or a composite $\mathsf{S/H}$, $\mathsf{H/P}$, $\mathsf{S/P}$ recording a claim whose warrant spans the indicated levels. Write $\mathrm{worst}(\sigma)$ for the largest (worst) warrant in σ. Two operations act on statuses:

- the *reported status of a single regime* is the union (range) of the warrants of its constituent dictionary entries, from the best to the worst present;
- the *composition of claims across regimes* collapses to the single worst warrant,

$$\mathrm{compose}(\sigma_1, \ldots, \sigma_k) \;=\; \max\{\mathrm{worst}(\sigma_1), \ldots, \mathrm{worst}(\sigma_k)\},$$

  matching the reference calculus, in which $\mathrm{compose}(\mathsf{S}, \mathsf{S/H}) = \mathsf{H}$, $\mathrm{compose}(\mathsf{S/H}, \mathsf{H/P}) = \mathsf{P}$.

In particular S is a unit for compose ($\mathrm{compose}(\mathsf{S}, \sigma) = \mathrm{worst}(\sigma)$) and compose is monotone: a chain is never more warranted than its weakest link. The two operations are genuinely different — $\mathrm{compose}(\mathsf{S/H}, \mathsf{S/H}) = \mathsf{H}$, whereas a single regime whose entries range over S to H reports $\mathsf{S/H}$ — and conflating them is a common way to overstate warrant, which is exactly what the calculus is built to prevent.

Proposition 7.2 (Monotonicity and unit)**.** **[S]** *With the order* $\mathsf{S} < \mathsf{H} < \mathsf{P}$ *and*

$$\mathrm{compose}(\sigma, \tau) = \max\{\mathrm{worst}(\sigma), \mathrm{worst}(\tau)\},$$

composition is associative and commutative, S *is a two-sided unit on the worst component, and for all* σ, τ *one has*

$$\mathrm{worst}(\sigma) \;\leq\; \mathrm{worst}(\mathrm{compose}(\sigma, \tau)).$$

Hence no composite claim can carry a better (smaller) worst warrant than any of its inputs.

Proof. max on the totally ordered warrant set $\{\mathsf{S} < \mathsf{H} < \mathsf{P}\}$ is associative and commutative, has the least element S as unit, and satisfies $w \leq \max(w, w')$. Since compose is max applied to the worst components, these properties transfer verbatim. □

This is a finite, checkable structure; it is implemented and property-tested in the accompanying Haskell development (section 8), where monotonicity, the unit law, and associativity are the checkable properties.

7.2 This regime's status, and its compositions

Proposition 7.3 (Status of the classical geometry regime)**. [S/H]** *The classical geometry regime's reported status is the range* $\mathsf{S/H}$*. Its internal entries are: Lorentzian manifold with metric,* $\mathsf{S/H}$ *(classical/semiclassical, not itself a quantum state); diffeomorphism groupoid of fields,* S *(a theorem-level structure, provided physical Killing symmetry is not conflated with gauge redundancy); moduli stack of Einstein solutions,* $\mathsf{S/H}$ *(rigorous where constructed, globally model-dependent); connection reformulation,* $\mathsf{S/H}$ *(exact on the nondegenerate sector, formulation-dependent across it). The range of these warrants runs from* S *to* H*, so the reported regime status is* $\mathsf{S/H}$*.*

Proof. By theorem 7.1 the reported status of a single regime is the union of the warrants of its entries. The entries contribute warrants $\{\mathsf{S},\mathsf{H}\} \cup \{\mathsf{S}\} \cup \{\mathsf{S},\mathsf{H}\} \cup \{\mathsf{S},\mathsf{H}\} = \{\mathsf{S},\mathsf{H}\}$, whose range is $\mathsf{S/H}$. Note this is the reported-range operation, *not* compose: the cross-regime composition of the same entries would collapse to worst $= \mathsf{H}$. □

We record three representative compositions with neighbouring regimes; each is a statement about warrant, not about existence of the bridge, and each uses compose (worst-component-wins), not the reported-range operation.

- **Classical geometry ∘ gauge/BRST regime.** $\mathrm{compose}(\mathsf{S/H}, \mathsf{S/H}) = \mathsf{H}$. This is the best-warranted *bridge* available to the classical regime — the perturbative BRST/BV observable algebra built on the classical solution stack (theorem 5.3) — yet it already lands at H, because the worst component of each S/H input is H. Only the pure-S structural sub-claims of the two regimes (the groupoid/isotropy identification of theorem 3.3; the derived-critical-locus construction where rigorous) compose back to S. This is the calculus's core discipline: one composes the specific claims used, and two $\mathsf{S/H}$ regimes do not yield an $\mathsf{S/H}$ bridge.

- **Classical geometry ∘ spin-network regime.** $\mathrm{compose}(\mathsf{S/H}, \mathsf{S/H}) = \mathsf{H}$, and the specific *bridging claim* that a large-spin/coherent-state limit of a spin network reproduces a given classical 3-geometry is independently only heuristic, consistent with H.

Regime	Role	Status
1. classical geometry (this paper)	Lorentzian solution stack	S/H
2. gauge / BV–BRST / derived	observables as cohomology	S/H
3. spin networks	quantum spatial geometry	S/H
4. spin foams	covariant histories	H
5. tensor networks / QEC	holographic encoding	H
6. positive geometries	amplitude boundaries	S/H
7. double copy	gravity as gauge^2	S/H
8. celestial holography	flat-space S-matrix	H
9. asymptotic safety	UV fixed point	H
10. factorization algebras	local-to-global observables	S/H
11. noncommutative geometry	spectral reconstruction	H
12. quantum causal structure	indefinite causal order	H/P

Table 2: The twelve regimes and their standalone statuses. The weakest-link composite of *all twelve* is P, driven by regime 12; this is the precise sense in which the program is modular rather than a single warranted theory (theorem 7.2).

- **Classical geometry ∘ causal-order (indefinite) regime.** $\mathrm{compose}(\mathsf{S/H}, \mathsf{H/P}) = \mathsf{P}$. Chaining the classical regime to a claim that presupposes genuinely indefinite (superposed) causal order pulls the composite down to speculative, no matter how careful the classical part is. This is monotonicity doing its intended work.

7.3 What "reconstruction" means for this regime, made precise

We can now state the reconstruction thesis for the classical regime as a checkable requirement rather than a slogan. A candidate deeper theory $\mathcal{T}$ (regimes 3–12) *reconstructs* the classical geometry regime in a limit $\lambda \to \lambda_0$ if there is a map

$$\mathrm{Real}_\lambda : \{\text{states/histories of } \mathcal{T}\} \longrightarrow \mathrm{Sol(Ein)} /\!\!/ \mathrm{Diff}(M)$$

into the solution *stack* — not the coarse quotient — such that, for symmetric target solutions, Real_λ reproduces the automorphism group $\mathrm{Isom}(M, g)$ of theorem 3.3 and the isotropy stratification of table 1. The requirement to land in the stack, with its automorphism data, is what makes "reproduces the classical limit" falsifiable: a theory that matches the coarse geometry but not the isotropy has not reconstructed the classical regime, it has reconstructed a lossy image of it. The status of any such reconstruction claim is bounded below by $\mathrm{compose}(\mathsf{S/H}, \mathrm{status}(\mathcal{T}))$, which for the nonperturbative regimes 4,5,8,9,11 is H and for regime 12 is P.

8 A machine-checkable model

The accompanying Haskell development models three checkable pieces of this paper: the status calculus of theorem 7.1 (with monotonicity, unit, and associativity as tested properties), the isotropy table of table 1 (as a lookup with dimension counts matching the Killing-field computations of section 4), and the automorphism identification of theorem 3.3 (as an action-groupoid whose automorphism group at an object is its stabilizer, with the isotropy-jump example of theorem 4.6 as a concrete instance). The core checkable property is that compose is monotone and has S as unit, matching theorem 7.2; a second checkable property is that the stabilizer computed from the modelled Diff-action on the discrete example agrees with the tabulated $\dim \mathrm{Kill}$. The development compiles with `ghc` and runs a demonstration that prints the isotropy of each tabulated solution, verifies the isotropy jump along the Kerr family, and reports the weakest-link status of a few sample regime chains. We stress that this is a model of the paper's finite combinatorial content (the status calculus and the isotropy bookkeeping), not a formalization of the analytic theorems of section 3, which remain at the level of cited results.

9 Limitations and open problems

We are explicit about what is not settled, because the epistemic labels are only meaningful if the caveats behind them are stated.

Non-properness of the Lorentzian Diff-action. The clean slice theorem (theorem 3.4) uses properness of the $\mathrm{Diff}(M)$-action on metrics, which holds for Riemannian metrics on a closed manifold. For Lorentzian metrics, on non-compact M, or with matter, the action is generally not proper, orbits need not be closed, and a global slice theorem in the same generality is not available. The action groupoid $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$ and the isotropy identification theorem 3.3 survive as groupoid statements regardless — this is one of the reasons the stack, which does not require the quotient to be a nice space, is the right object — but the clean local-model statements of theorems 3.4 and 3.5 carry their compact-CMC hypotheses and should not be overclaimed for the asymptotically flat black-hole examples of section 4.

No global 4D derived stack of solutions. Berktav's construction [20] of the moduli stack of vacuum Einstein solutions is explicit, but its sharpest results (the equivalence with a gauge theory) are special to three dimensions, where gravity is a Chern–Simons/BF-type theory. A fully general *global*, four-dimensional *derived* stack of vacuum solutions — handling non-generic isotropy jumps, non-compact Diff, and matter coupling in one construction — is not established in the literature as of this writing. The graded-Lie/L_∞ picture (theorem 5.2, [19]) gives the correct formal/perturbative tangent complex but not a global geometric object; completing it is the "homological PDE" program, which is open. This is precisely roadmap open-problem item 4 ("model the diffeomorphism quotient and isotropy for concrete solution families"), and we have discharged its *local/isotropy* half (section 4) while leaving the global-derived-stack half open.

Signature. Most sharp moduli and deformation results (slice theorem, Einstein-metric moduli in the sense of Besse [14]) are cleanest in Riemannian signature; the Lorentzian causal theory (theorems 2.4 and 2.7) and the moduli/deformation theory live in partly different technical worlds. We have kept the causal layer (section 2) and the moduli layer (sections 3 and 5) labelled separately rather than silently identifying them across a Wick rotation.

Physical versus gauge symmetry. The single conceptual point on which the whole construction turns: Killing symmetries are physical residual symmetries (they source conserved charges and degeneracies), not gauge redundancies. In the stack, gauge redundancy is the groupoid's morphisms and isotropy is an object's automorphism group; these are different roles for what look superficially like "diffeomorphisms." Conflating them — treating a Killing field as "pure gauge" — is exactly the error the stack exists to prevent, and it is the error a coarse quotient silently commits.

10 Conclusion

The classical regime of general relativity has, in the reconstruction program, a narrow and precise job: to package its content so that "reproduce the classical limit" is a falsifiable requirement on any deeper theory. We have argued that the package is the solution stack $\mathrm{Sol}(\mathrm{Ein})/\!\!/\mathrm{Diff}(M)$, an action groupoid whose objects are Einstein solutions, whose morphisms are diffeomorphisms, and whose automorphism group at a solution g is its isometry group $\mathrm{Isom}(M, g)$ (theorem 3.3). The choice of the groupoid over the coarse quotient is forced, not stylistic: by the slice theorem and the linearization-stability theorems of Moncrief, Fischer, Marsden and Arms, the coarse quotient $\mathrm{Sol}(\mathrm{Ein})/\mathrm{Diff}(M)$ is genuinely singular at exactly the symmetric solutions, and it is singular there because it has discarded the isotropy that the stack keeps (theorems 3.5 and 3.6). We made the isotropy concrete for the standard families and exhibited the Kerr-to-Schwarzschild isotropy jump $\mathbb{R} \times U(1) \rightsquigarrow \mathbb{R} \times SO(3)$ as a stratification of the stack (theorem 4.6 and table 1), and we identified the deformation complex whose H^0 is the Killing algebra and whose quadratic obstruction is the same data seen deformation-theoretically (theorems 5.1 and 5.2). We placed the regime in a modular, weakest-link status calculus (theorems 7.1 and 7.2) that bounds honestly how it composes with the eleven other regimes and never inflates a chain above its weakest link. And we stated what remains open: a global four-dimensional derived stack of solutions with non-proper Diff and matter is not available, and the reconstruction of this regime from any nonperturbative deeper theory is, by the calculus, at best a heuristic (H) claim until such a construction and such a limit are made precise.

The one sentence to carry forward is the reconstruction requirement made checkable in section 7: a deeper theory reconstructs classical geometry only if its semiclassical limit lands in the solution *stack*, reproducing the isotropy groups and their stratification — not merely in the coarse quotient, which has already thrown that data away.

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