

Observables from Gluing: Factorization Algebras, Cosheaves, and the Local-to-Global Reconstruction of Quantum Observables

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Abstract

We give a self-contained account of the local-to-global structure of quantum observables, organized around a single invariant: the way the observable algebra of a large region is assembled from the observable algebras of small regions by a gluing (colimit) law rather than by restriction. The technical vehicles are the factorization algebras of Costello and Gwilliam and the nets of algebraic quantum field theory in the Haag–Kastler and Brunetti–Fredenhagen–Verch traditions. We develop the definitions from scratch: prefactorization algebras as algebras over the colored operad of disjoint opens, Weiss covers, and the factorization (cosheaf-for-the-Weiss-topology) condition that distinguishes an observable assignment from a sheaf. Our central self-contained result is a reconstruction theorem: a locally constant, unital prefactorization algebra on the real line carries a canonical associative product, and the associativity of that product is forced by the associativity of the factorization structure maps for three disjoint intervals. We prove this for a finite combinatorial model in which every step is an exhaustive computation, and we provide accompanying Haskell code that checks associativity, unitality, and the weakest-link status calculus exactly on that model. We then state, with proof sketches and explicit references, the higher-dimensional statement (locally constant factorization algebras on $\mathbb{R}^n$ are E_n -algebras), the free-field construction (whose cohomology on the line recovers the Weyl algebra), and the comparison theorems relating factorization algebras to AQFT nets for free theories (Gwilliam–Rejzner; Benini–Perin–Schenkel; Benini–Musante–Schenkel). Each claim is tagged with an explicit epistemic label from the project’s S/H/P calculus: the reconstruction, E_n , free-field, and comparison statements are standard mathematics; the reading of renormalizability as the vanishing of a cohomological quantization obstruction is heuristic; and any background-independent, nonperturbative quantum-gravity factorization

algebra is, at present, speculative and open. We close with an honest account of the Lorentzian-signature, interacting, and background-dependence limitations, and with the open problems that separate the current state of the art from a genuinely gravitational local-to-global observable theory.

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1 Introduction

1.1 The reconstruction thesis for observables

This paper is one entry in a modular program whose governing perspective is that quantum gravity is less the quantization of matter placed inside a fixed spacetime than the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures. Each regime in the program isolates one such invariant. The regime treated here isolates the invariant that governs *measurement*: not a single global algebra of observables posited at the outset, but a rule that assigns to every region of spacetime the observables one can measure inside it, together with the compositional law by which measurements in small regions combine into measurements in large regions.

The mathematical content of that law is captured by two closely related structures. The first is the factorization algebra of Costello and Gwilliam [1, 2], which assigns to each open set U of a manifold a cochain complex $\mathcal{F}(U)$ of observables and, to each finite family of pairwise disjoint opens $U_1, \dots, U_n$ inside a larger open V , a structure map

$$\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V), \quad (1)$$

subject to associativity, symmetry, and a local-to-global gluing axiom formulated over a distinguished class of covers. The second is the net of algebras of algebraic quantum field theory (AQFT) in the Haag–Kastler tradition [3] and its locally covariant refinement due to Brunetti, Fredenhagen, and Verch [4], which assigns to each spacetime region a C^* - or $*$ -algebra $\mathcal{A}(U)$ subject to isotony, Einstein causality, and (in the covariant version) functoriality across all globally hyperbolic spacetimes at once.

The structural point that organizes the paper is the contrast in Equation (1) with the way a *sheaf* behaves. A sheaf reconstructs a global section from local sections by a limit: sections that agree on overlaps glue to a unique global section, so the global datum is a constraint satisfied by local restrictions. A factorization algebra reconstructs its global value by a colimit: the value on a large region is built up out of, and generated by, the values on small regions, glued by the composition maps (1). Observables compose; they do not merely restrict. This is the precise sense in which the observable content of a field theory is reconstructed local-to-global.

1.2 What is proved here, and what is cited

The literature on factorization algebras is substantial and technical, resting on homotopical algebra, functional analysis, and the theory of ∞ -categories. Rather than reproduce that machinery, this paper does three things.

First, we assemble a careful, notation-complete account of the definitions (Section 2): the colored operad of disjoint opens, prefactorization algebras, Weiss covers, the factorization/cosheaf condition, the descent contrast between colimits and limits, the relation to E_n -algebras, and the AQFT-net axioms. This section states the deep theorems (the E_n recognition principle, the free-field construction, the AQFT comparison theorems) with proof sketches and explicit citations, and it is honest about which parts are being cited rather than proved.

Second, we prove in full a finite, self-contained reconstruction theorem (Theorem 3.9 in Section 3). A locally constant, unital prefactorization algebra on the real line determines an associative product, and the associativity of that product is a consequence of the associativity of the structure maps for three disjoint intervals. We formulate this for a finite combinatorial model of intervals in which the statement becomes an exhaustive finite computation, so that the proof is complete and machine-checkable. This is the paper’s flagship illustration of the reconstruction thesis: an associative algebra — the prototype of a one-dimensional observable algebra — is *reconstructed* from purely local gluing data.

Third, we exhibit accompanying Haskell code (Section 4.4) implementing the finite model. The code builds the poset of intervals, the structure maps, and the recovered product, and it verifies associativity, unitality, and the weakest-link status calculus by exhaustive evaluation over a finite basis. A passing check is therefore a proof for the finite model, not a random sample.

1.3 Epistemic labels

Throughout, claims are tagged with one of three warrants drawn from the project’s status calculus (Section 3.1): [S] for a standard mathematical or mathematical-physics statement (a proved theorem or an established correspondence), [H] for a strong heuristic physical reading, and [P] for a speculative ontological extension. Composite labels such as [S/H] record that a claim mixes warrants. The labelling is not decorative. It is governed by a monotone composition rule (worst warrant wins), so that chaining regimes together can never raise the epistemic status of a conclusion above that of its weakest input. The reconstruction theorem, the E_n recognition principle, the free-field construction, and the factorization-algebra/AQFT comparison theorems are all [S]. The reading of renormalizability and anomaly-freedom as the vanishing of a cohomological quantization obstruction is [H]. The existence of a genuinely background-independent, nonperturbative quantum-gravity factorization algebra is [P] and open.

1.4 Organization

Section 2 develops the framework. Section 3 states and proves the main results. Section 4 works the one-dimensional example in detail, including the free particle and a locality/gluing consistency check, and describes the Haskell model. Section 5 places the regime among its neighbours in the program and runs the weakest-link calculus explicitly. Section 6 records limitations and open problems. Section 7 concludes.

2 Mathematical framework

Fix a ground field $\mathbb{k}$ of characteristic zero; in the physical applications $\mathbb{k} = \mathbb{C}$. We write $\mathbf{Vect}$ for the category of $\mathbb{k}$ -vector spaces and $\mathbf{Ch}$ for the category of cochain complexes of $\mathbb{k}$ -vector spaces, each a symmetric monoidal category under the tensor product over $\mathbb{k}$. Let M be a smooth manifold and $\mathbf{Op}(M)$ its poset of open subsets, ordered by inclusion.

2.1 Disjoint opens and the factorization operad

The combinatorics of (1) is that of a colored operad. Recall that a (symmetric, colored) operad $\mathcal{O}$ consists of a set of colors, sets of operations $\mathcal{O}(c_1, \dots, c_n; c)$ with inputs colored $c_1, \dots, c_n$ and output colored c , an action of the symmetric group Σ_n permuting inputs, identity operations, and an associative, equivariant composition.

Definition 2.1 (The colored operad of disjoint opens). Let Disj_M be the colored operad whose colors are the open subsets of M and whose operations are

$$\text{Disj}_M(U_1, \dots, U_n; V) = \begin{cases} \{*\} & \text{if the } U_i \text{ are pairwise disjoint and } U_i \subseteq V \text{ for all } i, \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition is the evident nesting of disjoint families; the symmetric group acts by permuting the input opens.

An operation exists exactly when the inputs sit disjointly inside the output. This encodes the geometric fact that observables supported in disjoint regions can be multiplied together and the product lands in any region containing all of them.

2.2 Prefactorization algebras

Definition 2.2 (Prefactorization algebra). Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. A *prefactorization algebra* on M valued in $\mathcal{C}$ is an algebra over the operad Disj_M in $\mathcal{C}$. Unwound, it is:

1. an object $\mathcal{F}(U) \in \mathcal{C}$ for each open $U \subseteq M$;
2. for each finite family of pairwise disjoint opens $U_1, \dots, U_n$ contained in an open V , a *structure map*

$$m_{U_1, \dots, U_n}^V : \mathcal{F}(U_1) \otimes \dots \otimes \mathcal{F}(U_n) \longrightarrow \mathcal{F}(V);$$

3. a unit map $\mathbf{1} \rightarrow \mathcal{F}(\emptyset)$,

subject to: (a) *equivariance*, $m_{U_{\sigma(1)}, \dots, U_{\sigma(n)}}^V$ equals $m_{U_1, \dots, U_n}^V$ precomposed with the symmetry of $\otimes$, for $\sigma \in \Sigma_n$; (b) *associativity/compatibility*, for nested disjoint families the two ways of composing structure maps agree, i.e. if each U_i is a disjoint union of $U_{i1}, \dots, U_{ik_i}$ inside U_i then

$$m_{U_1, \dots, U_n}^V \circ \bigotimes_i m_{U_{i1}, \dots, U_{ik_i}}^{U_i} = m_{U_{11}, \dots, U_{nk_n}}^V;$$

(c) *unitality*, the map $m_U^V : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ attached to a single open $U \subseteq V$ is functorial in the inclusion and, together with the unit, makes $\mathcal{F}(\emptyset)$ a unit object.

For $\mathcal{C} = \text{Vect}$ or Ch this is the setting of Costello and Gwilliam [1, Ch. 3]. The single-open maps m_U^V make $\mathcal{F}$ a covariant functor $\text{Op}(M) \rightarrow \mathcal{C}$; the multi-input maps make it much more.

Remark 2.3 (Locality of the maps, not of the values). There is no requirement that $\mathcal{F}(U)$ be small or local in any sense; the locality is carried by the *maps*. A single global vector space equipped with the multiplication of a commutative algebra is a (rather degenerate) prefactorization algebra. The interesting condition, isolating those prefactorization algebras that deserve to be called observables, is the gluing axiom of Section 2.3.

2.3 Weiss covers and the factorization condition

An ordinary open cover is too coarse to detect the compositional structure of observables, because it does not see how finitely many points are jointly distributed. The right notion is due to Weiss.

Definition 2.4 (Weiss cover). A collection $\mathfrak{U} = \{U_i\}_{i \in I}$ of opens of U is a *Weiss cover* of U if for every finite subset $S \subseteq U$ there is an index i with $S \subseteq U_i$.

A Weiss cover is in particular an open cover (take $|S| = 1$), but it is far finer: it must contain, for every k , opens large enough to hold any k chosen points. The collection of all opens diffeomorphic to a disjoint union of finitely many balls is the prototypical Weiss cover of a manifold. Weiss covers are the covering families of a Grothendieck topology on $\text{Op}(M)$, the *Weiss topology*.

Given a prefactorization algebra $\mathcal{F}$ and a Weiss cover $\mathfrak{U}$, the structure maps assemble into a simplicial (Čech-type) object whose p -th term records observables on p -fold “intersections” in the factorization sense,

$$\check{C}(\mathfrak{U}, \mathcal{F})_p = \bigoplus_{i_0, \dots, i_p} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_p}), \quad (2)$$

with the structure and restriction maps as faces. Its (normalized, totalized) realization is the *factorization Čech complex*, and there is a canonical map $\check{C}(\mathfrak{U}, \mathcal{F}) \rightarrow \mathcal{F}(U)$ induced by the structure maps.

Definition 2.5 (Factorization algebra). A prefactorization algebra $\mathcal{F}$ valued in Ch (or in a symmetric monoidal category with a good notion of homotopy colimit) is a *factorization algebra* if it is a cosheaf for the Weiss topology: for every open U and every Weiss cover $\mathfrak{U}$ of U , the canonical map

$$\check{C}(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} \mathcal{F}(U) \quad (3)$$

is a quasi-isomorphism. Equivalently, $\mathcal{F}(U)$ is the homotopy colimit of the values of $\mathcal{F}$ over the factorizing cover.

The equivalence (3) is the local-to-global axiom. It says the global observables on U are generated, up to coherent homotopy, by observables supported in the members of any factorizing cover, with the generation carried out by the composition maps (1). In the physical reading, a measurement on U is a homotopy colimit — a coherent gluing — of measurements localized in smaller regions.

Remark 2.6 (Cosheaf, not sheaf). Replacing the colimit in (3) by a limit, and the direct sum in (2) by a product, gives the defining condition of a (homotopy) sheaf. The two conditions are genuinely different, and observables satisfy the colimit one. The reason is physical: enlarging a region *adds* the observables it now contains and lets previously separated observables be multiplied, an operation of assembly (colimit), whereas a section of a sheaf on a large region is *constrained* to restrict correctly (limit). The distinction is the technical heart of the reconstruction thesis of Section 1.1.

2.4 Descent: colimits against limits

It is worth isolating the descent statement in elementary terms, because it is the formal shadow of the physics. Let $U = U_1 \cup U_2$ with U_1, U_2 open and suppose $\{U_1, U_2\}$ together with $U_1 \cap U_2$ generate a factorizing cover (in dimension one this holds for two overlapping intervals). For a factorization algebra valued in $\mathbf{Ch}$, (3) specializes to an exact triangle

$$\mathcal{F}(U_1 \cap U_2) \longrightarrow \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \longrightarrow \mathcal{F}(U) \xrightarrow{[1]}, \quad (4)$$

the factorization analogue of a Mayer–Vietoris sequence, with the arrows built from the structure maps rather than from restrictions. In cohomology this expresses $H^\bullet \mathcal{F}(U)$ as a colimit of the local pieces. For a sheaf one would instead obtain the dual triangle with a product in the middle and the map reversed; the direction of the arrows is exactly what distinguishes the two regimes.

2.5 Relation to E_n -algebras

On Euclidean space the factorization condition has a purely algebraic shadow. Call a factorization algebra $\mathcal{F}$ on $\mathbb{R}^n$ *locally constant* if for every inclusion of opens $D \subseteq D'$ each diffeomorphic to a ball, the structure map $\mathcal{F}(D) \rightarrow \mathcal{F}(D')$ is an equivalence.

Theorem 2.7 (E_n recognition; Lurie, Costello–Gwilliam). [S] *The category of locally constant factorization algebras on $\mathbb{R}^n$ valued in a symmetric monoidal ∞ -category $\mathcal{C}$ is equivalent to the category of E_n -algebras in $\mathcal{C}$. For $n = 1$ this specializes to an equivalence with associative (A_∞) algebras.*

Reference and idea. This is [13, §5.5.4] (there phrased for the equivalent notion of a locally constant algebra over the Ran space / topological chiral homology) and is developed in the factorization-algebra language in [1, Ch. 5, esp. §5.3]. The idea is that a ball’s worth of observables $A = \mathcal{F}(D)$ is well defined up to canonical equivalence by local constancy, and the configuration space of k disjoint sub-balls inside a larger ball is homotopy equivalent to the space of k -ary operations of the little n -disks operad E_n . Reading the structure maps through this equivalence equips A with an E_n -algebra structure, and the construction is an equivalence of categories. For $n = 1$ the little intervals operad is homotopy equivalent to the associative operad, recovering an A_∞ -algebra whose homotopy category of modules and Hochschild invariants match the factorization-homology invariants of the line. $\square$

Theorem 2.7 is the conceptual template for the finite reconstruction theorem of Section 3: it says that in one dimension the entire factorization structure is equivalent to an associative algebra, so recovering an associative product from one-dimensional gluing data is not an accident but the shadow of an equivalence of categories.

2.6 AQFT nets and locally covariant field theory

The AQFT tradition packages the same physics with a different emphasis: unital C^* - or $*$ -algebras, Lorentzian signature, and causality built in as a primitive axiom.

Definition 2.8 (Haag–Kastler net). Let (M, g) be a globally hyperbolic spacetime and let $\mathcal{K}(M)$ be its poset of relatively compact, causally convex open subregions. A *Haag–Kastler net* is a functor $\mathcal{A}: \mathcal{K}(M) \rightarrow C^*\text{-Alg}$ into unital C^* -algebras with injective unital $*$ -homomorphisms as morphisms, satisfying:

1. *isotony*: $U \subseteq V$ induces a unital inclusion $\mathcal{A}(U) \hookrightarrow \mathcal{A}(V)$;
2. *Einstein causality*: if U_1 and U_2 are spacelike separated then $[\mathcal{A}(U_1), \mathcal{A}(U_2)] = 0$ inside $\mathcal{A}(V)$ for any $V \supseteq U_1 \cup U_2$;
3. *time-slice*: if U contains a Cauchy surface of V then $\mathcal{A}(U) \rightarrow \mathcal{A}(V)$ is an isomorphism.

Definition 2.9 (Locally covariant QFT; Brunetti–Fredenhagen–Verch). Let $\mathbf{Loc}$ be the category whose objects are oriented, time-oriented, globally hyperbolic spacetimes and whose morphisms are isometric embeddings preserving orientations and time-orientations with causally convex image. A *locally covariant quantum field theory* is a functor $\mathcal{A}: \mathbf{Loc} \rightarrow C^*\text{-Alg}$ sending each morphism to a unital injective $*$ -homomorphism, satisfying Einstein causality and the time-slice axiom formulated across $\mathbf{Loc}$ [4, 5].

The move from Theorem 2.8 to Theorem 2.9 is the AQFT counterpart of the reconstruction thesis: the theory is defined not on subregions of one fixed spacetime but functorially on the category of all admissible spacetimes, so that which spacetime one works on becomes part of the data glued over rather than a fixed background. This is the sense in which locally covariant AQFT builds in a form of background independence at the level of the site.

2.7 Observables of a free theory

The running example that makes both formalisms computable is a free (Gaussian) theory. We recall the construction in the factorization language [1, Ch. 4] in the form we need.

Let E be the sheaf of fields (sections of a graded vector bundle on M), with a free BV action whose linearized equations of motion are governed by a formally self-adjoint Green-hyperbolic operator P . Write $E_c(U)$ for compactly supported fields on U . The *classical observables* are

$$\text{Obs}^{\text{cl}}(U) = (\text{Sym}(E_c(U)^\vee[-1]), d), \quad (5)$$

the (completed) symmetric algebra on the shifted dual of compactly supported fields, with d the Chevalley–Eilenberg/BV differential encoding the equations of motion. The *quantum*

observables $\text{Obs}^q(U)$ are a deformation of (5) over $\mathbb{k}[[\hbar]]$ whose differential is $d + \hbar \Delta$ for the BV Laplacian Δ built from the propagator of P ; the structure maps implement the (normal-ordered) operator product. The following is the free-theory output we cite.

Proposition 2.10 (Free-field factorization algebra; Costello–Gwilliam). [S] *For a free BV theory with Green-hyperbolic P , the assignments $U \mapsto \text{Obs}^{\text{cl}}(U)$ and $U \mapsto \text{Obs}^q(U)$ are factorization algebras on M , and Obs^q is a flat deformation of Obs^{cl} . On $M = \mathbb{R}$ with P the free-particle operator, the cohomology of the quantum observables on an interval recovers the Weyl algebra of the pair (q, p) , with the factorization product inducing its associative multiplication.*

Reference. The construction and the flatness statement are [1, Ch. 4]; the quantum-mechanics computation identifying the cohomology on the line with the Weyl algebra is worked in [1, Ch. 4, quantum mechanics], and is the one-dimensional instance of Theorem 2.7 carrying Obs^q to an associative algebra whose isomorphism class is the Weyl algebra. We return to a finite model of this computation in Section 4.3. $\square$

2.8 Factorization homology and global invariants

There is a global counterpart to the local assignment $\mathcal{F}$: an operation that integrates a factorization algebra over a manifold to produce a single invariant, in the way ordinary homology integrates a local coefficient system. For a locally constant factorization algebra on $\mathbb{R}^n$, equivalently an E_n -algebra A by Theorem 2.7, and for a framed n -manifold M , the *factorization homology*

$$\int_M A = \text{colim}_{(\text{Disk}_n \downarrow M)} A \quad (6)$$

is the homotopy colimit of A over the category of framed embeddings of finite disjoint unions of disks into M [13, 12]. It is the value one would assign to “all of M ” by gluing the local E_n -structure, and it satisfies an Eilenberg–Steenrod-type characterization: it takes disjoint unions to tensor products and satisfies an excision (collar-gluing) axiom

$$\int_M A \simeq \int_{M_0} A \otimes_{\int_{\mathbb{R} \times P} A} \int_{M_1} A \quad (7)$$

whenever $M = M_0 \cup_{\mathbb{R} \times P} M_1$ is a decomposition along a collar $\mathbb{R} \times P$. Excision (7) is the global face of the same local-to-global principle carried locally by (3): a global invariant is assembled from the invariants of the pieces, glued over the collar, by a (relative) tensor product rather than a fibre product.

Two special cases anchor the picture. When A is the (suitably interpreted) free commutative algebra, $\int_M A$ reduces to a form of the ordinary homology of M ; this is the sense in which factorization homology generalizes singular homology. When $M = S^1$ and A is an associative (E_1) algebra, factorization homology computes the Hochschild homology,

$$\int_{S^1} A \simeq \text{HH}_\bullet(A), \quad (8)$$

the derived trace of A [13, §5.5.3]. The circle is the worldline of a particle wrapped in time, and (8) says the corresponding global observable is the trace of the algebra recovered on the

line — the partition function of the one-dimensional theory. This is the point of contact with the cobordism/TQFT reading of the program: factorization homology over a closed manifold plays the role of a state-sum partition function, computed by gluing local algebraic data.

We record the one-dimensional consequence for later use, since it connects directly to the reconstruction theorem of the next section.

Proposition 2.11 (Global invariant of the line’s algebra). [S] *Let $\mathcal{F}$ be a locally constant factorization algebra on $\mathbb{R}$ valued in $\mathbf{Ch}$, with recovered associative algebra A as in Theorem 2.7. Then the factorization homology of the induced E_1 -structure over the circle is the Hochschild homology $\mathrm{HH}_\bullet(A)$; in particular, when A is the Weyl algebra of Theorem 2.10, this global invariant is the Hochschild homology of the Weyl algebra, a one-dimensional space concentrated in the appropriate degree.*

Reference. The identification $\int_{S^1} A \simeq \mathrm{HH}_\bullet(A)$ for an associative algebra A is [13, §5.5.3]; see also [12]. The Hochschild homology of the Weyl algebra over a field of characteristic zero is one-dimensional (concentrated in top degree), the standard computation via its Koszul resolution as a homologically smooth algebra. (The Weyl algebra is homologically smooth but not proper: it is infinite-dimensional over $\mathbb{k}$.) $\square$

3 Main results

We now isolate the reconstruction phenomenon in a form that is fully proved rather than cited. The strategy is to pass to a finite combinatorial model of the line, in which local constancy becomes an exact identification and the homotopy colimit of Theorem 2.5 becomes an ordinary colimit computable by hand.

3.1 The S/H/P status calculus

We first record the composition rule that governs the epistemic labels, since one of our results is about it. Let $(\{S, H, P\}, \leq)$ be the totally ordered set with $S < H < P$, read as increasing speculativeness.

Definition 3.1 (Status and composition). A *status* is an element of $\{S, H, P\}$. The *composition* of two statuses is their maximum, $\sigma * \tau = \max(\sigma, \tau)$.

Remark 3.2 (Composite labels). A claim may be tagged by a nonempty multiset of warrants, written as a composite label such as $[S/H]$. Such a label records provenance — that the claim carries both an S -part and an H -part — and its *effective status* is the composition (maximum) of its components; thus $[S/H]$ has effective status H , and $[H/P]$ has effective status P . The calculus of Theorem 3.3 acts on effective statuses, so no generality is lost by regarding a composite label as its maximum whenever it enters a composition.

Proposition 3.3 (Weakest-link calculus). [S] *The operation $*$ of Theorem 3.1 is associative and commutative, has S as a two-sided unit, and is monotone: $\sigma \leq \sigma * \tau$ for all σ, τ . Consequently, any finite composite of statuses equals the maximum of its inputs, and no composition can produce a status strictly below every input.*

Proof. Maximum on a totally ordered set is associative and commutative, and the least element S is its unit. Monotonicity $\sigma \leq \max(\sigma, \tau)$ is immediate, and an induction on the number of inputs gives that a finite composite equals the maximum of the inputs; the maximum is never strictly below any input, in particular never below the smallest input, which is the weakest link. $\square$

Theorem 3.3 is elementary but load-bearing: it is the reason the program is modular rather than unified. Chaining the present regime to another can never launder a speculative input into a standard conclusion. We use it explicitly in Section 5, and it is one of the properties checked by the accompanying code.

3.2 A finite model of the line

Fix an integer $N \geq 1$ and let $L = \{1, 2, \dots, N\}$ be a finite ordered set of sites, a discretization of an interval of the real line. By an *interval* we mean a nonempty set of the form $[a, b] = \{a, a+1, \dots, b\}$ with $1 \leq a \leq b \leq N$; the empty set is also admitted as the empty interval $\emptyset$. Two intervals are *disjoint* if they share no site, and we say $[a, b]$ lies *to the left of* $[c, d]$ if $b < c$. By convention the empty interval $\emptyset$ is disjoint from every interval and is regarded as lying both to the left of and to the right of every interval; consistently with $\mathcal{F}(\emptyset) = \mathbf{1}$, it contributes the unit under the structure maps and its placement carries no ordering content. The ambient interval is $L = [1, N]$.

Definition 3.4 (Finite prefactorization algebra on L). Let $(\mathcal{C}, \otimes, \mathbf{1})$ be a symmetric monoidal category. A *finite prefactorization algebra* on L is an assignment $I \mapsto \mathcal{F}(I)$ of an object of $\mathcal{C}$ to each interval (with $\mathcal{F}(\emptyset) = \mathbf{1}$), together with structure maps

$$m_{I_1, \dots, I_n}^J: \mathcal{F}(I_1) \otimes \dots \otimes \mathcal{F}(I_n) \rightarrow \mathcal{F}(J)$$

for pairwise disjoint intervals $I_1, \dots, I_n \subseteq J$, satisfying the equivariance, associativity, and unitality axioms of Theorem 2.2 restricted to intervals.

Definition 3.5 (Locally constant, unital). A finite prefactorization algebra $\mathcal{F}$ on L is *locally constant* if for every inclusion of nonempty intervals $I \subseteq J$ the single-open structure map $m_I^J: \mathcal{F}(I) \rightarrow \mathcal{F}(J)$ is an isomorphism. It is *unital* if the unit map $u: \mathbf{1} = \mathcal{F}(\emptyset) \rightarrow \mathcal{F}(J)$ (the empty structure map into J) provides a two-sided unit for the binary products defined below.

Local constancy lets us transport every $\mathcal{F}(I)$ to a single object. Fix a nonempty interval and call the transported object $A := \mathcal{F}([1, 1])$; for any nonempty interval I we obtain a canonical isomorphism $\varphi_I: \mathcal{F}(I) \xrightarrow{\sim} A$ by composing the (invertible) single-open maps along a chain of inclusions $[1, 1] \subseteq \dots \subseteq I$ or their inverses. The associativity axiom makes φ_I independent of the chosen chain: any two chains of inclusions between the same two intervals give the same composite, because each elementary square of inclusions commutes by the compatibility axiom of Theorem 2.2.

Lemma 3.6 (Well-defined transport). [S] *For a locally constant finite prefactorization algebra on L , the isomorphisms $\varphi_I: \mathcal{F}(I) \xrightarrow{\sim} A$ are well defined (independent of the chain of inclusions used to build them) and compatible: for $I \subseteq J$ one has $\varphi_J \circ m_I^J = \varphi_I$.*

Proof. The poset of nonempty intervals under inclusion is directed upward (any two intervals contained in L are contained in L) and its inclusion maps are sent to isomorphisms. In such a diagram of isomorphisms over a connected poset, any two directed paths between the same objects compose to the same isomorphism, by the functoriality clause of Theorem 2.2(c): each commuting inclusion square yields the relation among the four maps, and connectedness propagates it. Hence φ_I depends only on I , and $\varphi_J \circ m_I^J = \varphi_I$ is the statement that transport is compatible with a single further inclusion, which holds by construction. $\square$

3.3 Reconstruction of an associative algebra

We now read off a binary product from the ordering of the line.

Construction 3.7 (The recovered product). Let $\mathcal{F}$ be a locally constant, unital finite prefactorization algebra on L with $N \geq 3$, valued in $\mathbf{Vect}$. Choose disjoint nonempty intervals I_1 to the left of I_2 , both contained in a nonempty interval J . Define

$$\mu: A \otimes A \rightarrow A, \quad \mu = \varphi_J \circ m_{I_1, I_2}^J \circ (\varphi_{I_1}^{-1} \otimes \varphi_{I_2}^{-1}). \quad (9)$$

Set $1_A := \varphi_J \circ u \in A$, the image of the unit.

Lemma 3.8 (Independence of choices). [S] *The product μ of (9) is independent of the choice of the left interval I_1 , the right interval I_2 , and the ambient J , provided I_1 lies to the left of I_2 and both lie in J .*

Proof. Suppose $I_1 \subseteq I'_1$, $I_2 \subseteq I'_2$ are two admissible left and right choices inside $J \subseteq J'$, still with I'_1 to the left of I'_2 . The associativity axiom of Theorem 2.2 gives the commuting square

$$\begin{array}{ccc} \mathcal{F}(I_1) \otimes \mathcal{F}(I_2) & \xrightarrow{m_{I_1, I_2}^J} & \mathcal{F}(J) \\ m_{I_1}^{I'_1} \otimes m_{I_2}^{I'_2} \downarrow & & \downarrow m_{J'}^{J'} \\ \mathcal{F}(I'_1) \otimes \mathcal{F}(I'_2) & \xrightarrow{m_{I'_1, I'_2}^{J'}} & \mathcal{F}(J') \end{array}$$

because both composites equal $m_{I_1, I_2}^{J'}$ by compatibility of nested disjoint families. Transporting all four corners to A via the isomorphisms $\varphi_{(-)}$, and using Theorem 3.6 to turn the vertical single-open maps into identities on A , collapses the square to the equation $\mu_{I_1, I_2, J} = \mu_{I'_1, I'_2, J'}$. Since any two admissible choices have a common enlargement of this form, μ is well defined. $\square$

Theorem 3.9 (Reconstruction of an associative unital algebra). [S] *Let $\mathcal{F}$ be a locally constant, unital finite prefactorization algebra on $L = [1, N]$ with $N \geq 3$, valued in $\mathbf{Vect}$. Then $(A, \mu, 1_A)$ of Theorem 3.7 is an associative unital algebra. Its multiplication and unit are determined by the factorization structure maps, and the associativity of μ is a consequence of the associativity axiom of Theorem 2.2 for three disjoint intervals.*

Proof. Unitality. Let I_2 be a nonempty interval and $I_1 = \emptyset$ the empty interval to its left inside J . The unital axiom of Theorem 3.5 says $m_{\emptyset, I_2}^J = m_{I_2}^J \circ (u \otimes \text{id})$ under the canonical identification $\mathbf{1} \otimes \mathcal{F}(I_2) \cong \mathcal{F}(I_2)$; transporting to A gives $\mu(1_A \otimes a) = a$. The mirror

argument with the empty interval to the right gives $\mu(a \otimes 1_A) = a$. Hence 1_A is a two-sided unit.

Associativity. Choose pairwise disjoint nonempty intervals I_1, I_2, I_3 inside J with I_1 to the left of I_2 and I_2 to the left of I_3 ; this is possible because $N \geq 3$ (take the single sites $I_1 = [1, 1]$, $I_2 = [2, 2]$, $I_3 = [3, 3]$ and $J = [1, 3]$). Let J_{12} be a nonempty interval with $I_1, I_2 \subseteq J_{12}$ and J_{12} to the left of I_3 (for the single-site choice, $J_{12} = [1, 2]$), and let J_{23} be a nonempty interval with $I_2, I_3 \subseteq J_{23}$ and J_{23} to the right of I_1 (for the single-site choice, $J_{23} = [2, 3]$); both lie inside J . The associativity axiom of Theorem 2.2 applied to the two nestings $\{I_1, I_2\} \subseteq J_{12}$, $\{J_{12}, I_3\} \subseteq J$ and $\{I_2, I_3\} \subseteq J_{23}$, $\{I_1, J_{23}\} \subseteq J$ gives

$$m_{J_{12}, I_3}^J \circ (m_{I_1, I_2}^{J_{12}} \otimes \text{id}) = m_{I_1, I_2, I_3}^J = m_{I_1, J_{23}}^J \circ (\text{id} \otimes m_{I_2, I_3}^{J_{23}}), \quad (10)$$

since both outer composites equal the single ternary structure map m_{I_1, I_2, I_3}^J by compatibility of nested disjoint families. Transport each object to A using Theorem 3.6; the single-open maps become identities, the binary maps become μ by Theorem 3.8, and (10) becomes

$$\mu \circ (\mu \otimes \text{id}) = \mu \circ (\text{id} \otimes \mu): A \otimes A \otimes A \rightarrow A,$$

which is associativity of μ . Commutativity is *not* claimed: the left/right ordering of intervals on the line breaks the symmetry, exactly as the operad E_1 is associative but not commutative. $\square$

Remark 3.10 (Why this is the reconstruction thesis in miniature). Theorem 3.9 manufactures the multiplication of an associative algebra — the prototypical one-dimensional observable algebra — out of nothing but the gluing maps of local data on a line, with associativity forced by the gluing compatibility for three disjoint regions. No global product is posited; it is reconstructed. This is the finite, fully proved shadow of Theorem 2.7 and of Theorem 2.10, and it is the statement the accompanying code verifies exactly.

3.4 Comparison with AQFT nets

We record the bridge theorems relating the two formalisms, which we cite rather than prove. They are the rigorous justification for treating factorization algebras and AQFT nets as two presentations of the same local-to-global observable content.

Theorem 3.11 (Nets and factorization algebras for free theories). [S] *The following comparison results hold for free field theories.*

1. (Gwilliam–Rejzner [7].) *For a free theory whose equations of motion are Green-hyperbolic, there is a natural transformation intertwining the Costello–Gwilliam factorization algebra of quantum observables and the perturbative AQFT net of the same theory, matching their observable content.*
2. (Benini–Perin–Schenkel [8].) *On globally hyperbolic Lorentzian manifolds there is an equivalence between Cauchy-constant additive algebraic quantum field theories and Cauchy-constant additive time-orderable prefactorization algebras, realized by explicit functors in both directions.*

3. (Benini–Musante–Schenkel [9].) *For a natural class of free BV theories on the category of m -dimensional globally hyperbolic Lorentzian manifolds, the two quantizations — as a time-orderable prefactorization algebra and as a cochain-valued AQFT — are related by an explicit isomorphism of time-orderable prefactorization algebras, constructed from retarded and advanced Green’s homotopies.*

References. Statement (1) is the main theorem of [7], proved with the free scalar as running example and valid for any Green-hyperbolic free theory. Statement (2) is the equivalence theorem of [8], established model-independently at the level of the relevant categories of AQFTs and time-orderable prefactorization algebras. Statement (3) is the comparison of [9], whose key device is the generalization of retarded/advanced Green’s operators to Green’s homotopies on cochain complexes of linear differential operators. $\square$

Remark 3.12 (Direction of the remaining gap). Theorem 3.11 covers free (Gaussian) theories and their linear BV extensions. The interacting case — where the structure maps carry the operator product expansion and the quantization is controlled by an obstruction theory — is not covered by a comparably clean equivalence, and the gauge-fixed gravitational case is open. We state the relevant obstruction result next, and we return to the gap in Section 6.

3.5 Quantization as a cohomological obstruction

The passage from classical to quantum observables is not automatic. In the factorization/BV formalism it is controlled by a sequence of cohomology classes.

Proposition 3.13 (Quantization obstruction; Costello–Gwilliam). [S] *for the mathematics, [H] for the physical reading. Let Obs^{cl} be the classical BV factorization algebra of a theory. Order by order in $\hbar$, the obstruction to extending a quantization Obs^{q} to the next order lies in a local cohomology group; if all obstruction classes vanish, a quantization exists, and the set of quantizations is a torsor over the group controlling the next degree. In the physical reading, this identifies renormalizability and anomaly-freeness with the vanishing of these cohomological obstructions.*

Reference and label. The obstruction-theoretic statement is developed in [14] and in [2, Ch. 7–9, Vol. 2]; the identification of the obstruction with a local (deformation-complex) cohomology class and the torsor description of the choices are theirs. The mathematical content is standard [S]. The identification of the vanishing of these classes with the physics of renormalizability and anomaly cancellation is a strong and well-supported heuristic reading rather than a theorem about nature, so the physical clause is labelled [H]. $\square$

4 Worked example: the line

We make Theorem 3.9 concrete, exhibit a locality/gluing check, connect to the free particle, and describe the accompanying code.

4.1 A group-algebra model

Take $\mathcal{C} = \text{Vect}$ over $\mathbb{k}$ and fix a finite group G . Define a finite prefactorization algebra on $L = [1, N]$ by

$$\mathcal{F}(I) = \mathbb{k}[G] \text{ for every nonempty interval } I, \quad \mathcal{F}(\emptyset) = \mathbb{k},$$

with single-open maps $m_I^J = \text{id}_{\mathbb{k}[G]}$ (so $\mathcal{F}$ is locally constant), unit $u: \mathbb{k} \rightarrow \mathbb{k}[G]$ the inclusion $1 \mapsto e$ of the identity element, and binary structure map, for I_1 to the left of I_2 ,

$$m_{I_1, I_2}^J(x \otimes y) = x \cdot y \quad (x, y \in \mathbb{k}[G]),$$

the convolution product of $\mathbb{k}[G]$, extended to n -ary maps by the ordered product $x_1 \cdots x_n$ following the left-to-right order of the intervals.

Proposition 4.1 (The group-algebra model is a valid finite pfa). [S] *The assignment above satisfies the equivariance, associativity, and unitality axioms of Theorem 3.4, is locally constant and unital in the sense of Theorem 3.5, and the recovered algebra $(A, \mu, 1_A)$ of Theorem 3.7 is isomorphic to the group algebra $\mathbb{k}[G]$ with its convolution product.*

Proof. Equivariance for n -ary maps is the statement that reordering the disjoint intervals reorders the product accordingly; since the ordered product uses the geometric left-to-right order, a permutation of the labels acts by the same permutation on the tensor factors, and both sides compute the same ordered product. Associativity of the structure maps reduces to associativity of the convolution product of $\mathbb{k}[G]$, which holds because group multiplication is associative. Unitality is the identity $e \cdot x = x = x \cdot e$. Local constancy is immediate since the single-open maps are identities. The recovered product μ of (9) is then convolution, and $1_A = e$, so $(A, \mu, 1_A) \cong \mathbb{k}[G]$. $\square$

Theorem 4.1 is a concrete witness of Theorem 3.9: the group algebra is reconstructed from the line, and its associativity is the associativity of the ternary structure map. For a nonabelian G the recovered algebra is noncommutative, which is why the left-to-right ordering of intervals is essential.

4.2 A locality/gluing consistency check

Local constancy and the structure maps must be mutually consistent: transporting a product computed in a small ambient interval up to a large one must agree with the product computed directly in the large one. This is the finite avatar of the factorization gluing axiom (3), and it is a nontrivial constraint that a careless assignment can fail.

Proposition 4.2 (Gluing consistency). [S] *For the group-algebra model, and for any locally constant unital finite prefactorization algebra, the following diagram commutes for disjoint I_1 (left) and I_2 (right) inside $J \subseteq J'$:*

$$\begin{array}{ccc} \mathcal{F}(I_1) \otimes \mathcal{F}(I_2) & \xrightarrow{m_{I_1, I_2}^J} & \mathcal{F}(J) \\ & \searrow m_{I_1, I_2}^{J'} & \downarrow m_J^{J'} \\ & & \mathcal{F}(J'). \end{array}$$

Equivalently, the recovered product μ is the same whether computed through J or through J' ; this is precisely Theorem 3.8.

Proof. The triangle is the associativity axiom of Theorem 2.2 for the nesting $\{I_1, I_2\} \subseteq J \subseteq J'$: both composites equal $m_{I_1, I_2}^{J'}$. For the group-algebra model this reads $x \cdot y$ computed in $\mathbb{k}[G]$ regardless of the ambient interval, which holds because the single-open maps are identities. The equivalence with Theorem 3.8 is the transport of the triangle to A . $\square$

The point of stating Theorem 4.2 separately is diagnostic: it is exactly the identity that fails if one tries to define the structure maps inconsistently (for example by letting the product depend on the length of the ambient interval), and it is the identity the code checks by exhaustive enumeration over G and over admissible interval configurations.

4.3 The free particle and the Weyl algebra

Theorem 2.10 identifies the cohomology of the quantum observables of the free particle on the line with the Weyl algebra. We indicate the finite shadow of this statement to connect the abstract construction with Theorem 3.9.

For the free particle the classical observables on an interval are generated by the value and derivative of the field at a point, i.e. by a position mode q and a momentum mode p , and the classical factorization product is commutative. The BV Laplacian Δ deforms the product; the deformation is measured by the propagator, and at first order in $\hbar$ it introduces the commutator

$$[q, p] = \mu(q \otimes p) - \mu(p \otimes q) = \hbar, \quad (11)$$

the canonical commutation relation. The associative algebra recovered by Theorem 2.7 on the line is thus the Weyl algebra $W = \mathbb{k}\langle q, p \rangle / ([q, p] - \hbar)$, and its associativity is the associativity manufactured in Theorem 3.9. The left-to-right ordering of intervals encodes the operator ordering, which is exactly why the recovered product is noncommutative even though the classical one was commutative: noncommutativity of W is the $\hbar$ -linear shadow of the ordering built into E_1 . The Weyl algebra itself admits no finite-dimensional model: over a field of characteristic zero it is simple, and taking the trace of $[q, p] = \hbar \mathbf{1}$ would force $0 = \dim(V) \hbar$, so it has no nonzero finite-dimensional representation. A finite model instead captures the *exponentiated* (Weyl-form) relations $UV = e^{2\pi i/k} VU$, whose finite solutions are the algebras of finite Heisenberg groups, that is finite-dimensional group algebras of the kind used in Section 4.1. It is to these finite group algebras — not to a quotient of the algebraic Weyl algebra, which does not exist — that Theorem 3.9 and the accompanying code apply verbatim.

4.4 The Haskell model

The accompanying code, in `src/factorization-algebras/`, implements the finite theory of Sections 3.2 and 4.1 and checks its claims by exhaustive computation. It is organized as follows.

- **Core.hs** defines intervals on $L = [1, N]$, disjointness and left-of ordering, the group $\mathbb{Z}/k$ and its group algebra over the rationals as association lists (avoiding the hidden **containers** package), the single-open and n -ary structure maps, local-constancy transport, and the recovered product μ of Theorem 3.7.
- **Properties.hs** states the checkable claims as deterministic, exhaustive tests: associativity of μ over all triples of basis elements (Theorem 3.9); left and right unitality; the gluing-consistency identity of Theorem 4.2 over all admissible interval configurations up to a cutoff; equivariance of the binary map under swap combined with the abelian case; and the weakest-link status calculus of Theorem 3.3 (associativity, commutativity, the unit law for S , and monotonicity) over all triples in $\{S, H, P\}$.
- **Main.hs** runs the demonstrations — printing the recovered product table for a small group, exhibiting the reconstruction of associativity, and running the full verification suite — and exits nonzero if any check fails.

Because every check ranges exhaustively over a finite basis and uses exact rational arithmetic, a passing run is a proof of the corresponding finite claim, not a sampled estimate. The code compiles with `ghc` using only the `base` package.

5 Relation to other regimes and the weakest-link calculus

The factorization/AQFT regime sits at a definite place in the modular program, and its relations to neighbouring regimes are governed by Theorem 3.3. We make three of them explicit.

5.1 Gauge and constraint structure (BV–BRST)

The immediate predecessor is the BV–BRST regime of gauge and constraint structure, whose invariant is that physical observables are the degree-zero cohomology $H^0(Q_{\text{BV}})$ of a gauge-resolving differential on a single background. The factorization regime globalizes this: the quantum observables $\text{Obs}^q(U)$ form a factorization algebra whose cohomology $H^0(\text{Obs}^q(U))$ recovers the BV–BRST gauge-invariant observables on the region U , now assembled local-to-global across M rather than computed once around a fixed background [2, Vol. 2]. Both the BV–BRST identification and the factorization packaging are [S/H]: standard mathematics for the perturbative/local statement, heuristic as claims about the observables of interacting quantum gravity. By Theorem 3.3 the composite is [S/H], unchanged; the two regimes reinforce without either laundering the other’s status.

5.2 Holographic encoding (tensor networks and QEC)

A second neighbour reconstructs measurement by redundant encoding: bulk observables represented on boundary subregions via quantum-error-correcting codes, with entanglement-wedge reconstruction as the dictionary. The local subalgebra structure of a factorization

algebra or AQFT net is the algebraic substrate on which such reconstruction statements are phrased, and the modular/Type III structure of the local algebras [15] is the natural meeting point. Entanglement-wedge reconstruction itself carries status [H] (a semiclassical holographic statement), so by Theorem 3.3 any composite claim built from the factorization regime *and* entanglement-wedge reconstruction is at best [H]. This is an honest ceiling: the algebraic side is [S/H], but the composite inherits the weaker holographic warrant.

5.3 Asymptotic boundary data (celestial holography)

A third neighbour reconstructs the S -matrix from conformal data on the celestial sphere and asymptotic (BMS) symmetry representation theory. This is a different local-to-global scheme — observables organized by null infinity rather than by a Weiss cover of a bulk region — and its overall warrant is [H]. Composing it with the factorization regime again yields [H] by Theorem 3.3. The three neighbours realize the same reconstruction thesis by structurally different means (compositional gluing here, redundant encoding there, asymptotic-symmetry representation theory in the third), and the weakest-link calculus keeps the composite honest in each case.

5.4 Background independence and the ceiling

The factorization-algebra assignment lives on a fixed manifold M ; the locally covariant AQFT refinement (Theorem 2.9) improves this to a functor over the category $\mathbf{Loc}$ of all globally hyperbolic spacetimes, which is background independence at the level of the site but not at the level of a fluctuating metric. A genuinely background-independent, nonperturbative quantum-gravity observable algebra — of the kind one would want to extract from a spin-foam state sum or an asymptotically safe fixed point — is not known to be a factorization algebra or a cosheaf at all. Any claim that packages such an object as a factorization algebra therefore composes a [P] (speculative, currently unconstructed) input with the [S/H] factorization machinery, and by Theorem 3.3 lands at [P]. This is the precise sense in which the reconstruction thesis is, in this regime, proved where the theory is free or perturbative and open where it is gravitational.

6 Limitations and open problems

We collect the limitations honestly, separating what is established from what is programmatic.

Lorentzian signature. The Costello–Gwilliam construction is developed most cleanly in Euclidean signature and in a formal ($\hbar$ -adic) perturbative sense; the propagator that defines the BV Laplacian is a Euclidean Green’s function. The Lorentzian theory requires the time-ordered/Green-hyperbolic technology of perturbative AQFT and of the time-orderable prefactorization algebras of Benini–Perin–Schenkel and Benini–Musante–Schenkel [8, 9]. The comparison theorems (Theorem 3.11) are the current bridge, and they are restricted to free (or linear BV) theories. Extending the equivalence to interacting Lorentzian theories is open.

Interacting and renormalized observables. For interacting theories the quantum factorization algebra exists only after renormalization, and its very existence is the content of the obstruction theory of Theorem 3.13. The structure maps then carry the operator product expansion, and there is no closed-form finite model of the kind we used in Section 4. The finite reconstruction theorem Theorem 3.9 applies to the associative (one-dimensional, or free) shadow, not to the full interacting higher-dimensional theory.

Background dependence. Even the locally covariant refinement fixes the class of space-times and treats the metric as background data of the site, not as a quantized field. Packaging a fluctuating-metric, background-independent observable algebra — the actual object of quantum gravity — as a factorization algebra is not achieved and is flagged [P].

Analytic subtleties. The homotopy colimits of Theorem 2.5 and the completed symmetric algebras of Equation (5) require care with functional-analytic completions (nuclear or convenient/bornological vector spaces) and with the Weiss topology’s non-standard covers. Our finite model deliberately sidesteps these by working with finite-dimensional algebras, where colimits are ordinary and completions are trivial; the price is that it captures the algebraic skeleton, not the analysis.

Open problems. Three problems mark the boundary of the current state of the art.

1. Extend the free-theory comparison (Theorem 3.11) to interacting, gauge-fixed (BV-quantized) perturbative gravity, matching factorization-algebra and AQFT observables order by order in $\hbar$.
2. Decide whether any background-independent observable algebra — from a spin-foam state sum, a loop-quantum-gravity kinematical setup, or an asymptotically safe fixed point — can be presented as a factorization algebra or a cosheaf of observables at all. This is not done, and should be treated as open rather than assumed.
3. Develop the modular-theoretic (Type III, relative-entropy) bridge between AQFT local algebras and the holographic-QEC reconstruction of Section 5.2, which is currently underexplored.

7 Conclusion

The observable content of a quantum field theory is not a single global algebra handed down at the outset. It is a rule assigning to each region the measurements possible there, together with a compositional law by which local measurements generate global ones. Factorization algebras make that law precise as a cosheaf condition over the Weiss topology, gluing by colimits (observables compose) rather than by limits (sections restrict); AQFT nets make it precise as a functor with built-in causality, and the two are provably the same for free theories. In one dimension the entire structure collapses to an associative algebra, and we proved, for a finite combinatorial model in which every step is an exhaustive computation, that the associative product and its associativity are *reconstructed* from the factorization

gluing data alone. The accompanying Haskell code checks that reconstruction, together with unitality, gluing consistency, and the weakest-link status calculus, exactly on the finite model.

Read against the program’s governing thesis, the regime supplies the measurement layer of the reconstruction of physics from invariant representation structures: it is the mechanism by which what can be measured is assembled from the compositional structure of local data. The honest boundary of the achievement is sharp. The reconstruction is a theorem for free and perturbative theories, a strong heuristic for the identification of renormalizability with a cohomological obstruction, and an open problem — flagged speculative and governed by the weakest-link calculus — for the genuinely background-independent, gravitational case.

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