

The Flat-Space S-Matrix as Conformal Data on the Celestial Sphere: Mellin Bases, Asymptotic Symmetries, and Soft Theorems as Ward Identities as the Boundary Layer of Reconstructed Spacetime

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Abstract

We give a self-contained account of celestial conformal field theory as a representation of the flat-space S -matrix, organized around a single epistemic thesis: quantum gravity in this regime is not the quantization of matter in a fixed spacetime but the reconstruction of the scattering process from invariant boundary data on the celestial sphere at null infinity. The boundary data are the asymptotic (Bondi–Metzner–Sachs) symmetries and the soft theorems, which are their Ward identities.

Three statements carry the account. First, the four-dimensional Lorentz group $SL(2, \mathbb{C})$ acts on the celestial 2-sphere as the global conformal group, so a massless S -matrix element written on boost eigenstates transforms as a two-dimensional conformal correlator. Second, the change of basis from plane-wave momentum eigenstates to boost eigenstates is a Mellin transform in the external energies; the resulting conformal primary wavefunctions of Pasterski and Shao form a complete basis on the principal continuous series $\Delta \in 1 + i\mathbb{R}$. Third, Weinberg’s leading soft-graviton theorem is equivalent to the Ward identity of a diagonal supertranslation subgroup of $BMS^+ \times BMS^-$, following He, Lysov, Mitra and Strominger, and the abelian soft-photon theorem is the Ward identity of large $U(1)$ gauge transformations, following He, Mitra, Porfyriadis and Strominger.

Each result carries the project’s $S/H/P$ epistemic tag. The Mellin change of basis, the $SL(2, \mathbb{C}) =$ conformal identification, and the leading soft-theorem/Ward-identity equivalence are standard mathematics and mathematical physics [S]; the celestial operator product expansion and the conformally soft current algebra are heuristically

established in tree-level and self-dual sectors [S/H]; and the claim that the celestial correlators assemble into a complete, intrinsically defined holographic dual of flat-space quantum gravity is heuristic at best and speculative in general [H/P]. We keep this status carefully distinct from AdS/CFT holography. We work two examples in full: the Mellin transform of a massless four-point contact amplitude, which exhibits the distributional support of celestial correlators on the locus where the cross-ratio is real; and the translation of Weinberg’s soft factor into celestial variables together with its identification with the supertranslation charge. An appendix documents accompanying Haskell code that checks the Mellin normalization, the Möbius group action and cross-ratio invariance, the weight bookkeeping $(h, \bar{h}) = ((\Delta + J)/2, (\Delta - J)/2)$, the Euler beta-function structure of the celestial gluon OPE, and the homogeneity of the soft factor.

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1 Introduction

1.1 The reconstruction thesis in the flat-space regime

A scattering amplitude in four-dimensional Minkowski space is ordinarily presented as a function of on-shell momenta, computed from a Lagrangian by a Feynman expansion or a recursion relation, and evaluated on plane-wave asymptotic states. In this presentation the spacetime is fixed background data and the amplitude is a derived object. Celestial conformal field theory reverses the order of explanation. It treats the S -matrix as a family of correlation functions on the two-sphere at null infinity, and it treats the symmetries of that boundary — the asymptotic Bondi–Metzner–Sachs (BMS) group and its soft-theorem Ward identities — as the invariant data from which the bulk scattering process is to be reconstructed.

This paper develops that reversal as an instance of a general thesis about quantum gravity that we state once and then use as an organizing principle: *quantum gravity is not primarily the quantization of material objects in a fixed spacetime; it is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures*. In the flat-space regime the invariant structure is the asymptotic symmetry algebra together with its Ward identities. The soft theorems are those Ward identities. The Mellin transform to boost eigenstates is the change of basis that makes the boundary conformal structure manifest. The celestial correlators are the reconstructed S -matrix, now presented as conformal data.

We are careful throughout to separate what is established from what is programmatic. The identification of the four-dimensional Lorentz group with the global conformal group of the celestial sphere is a statement about the action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{CP}^1$, and it is exact. The Mellin transform is an invertible change of basis on the principal series. The equivalence between Weinberg’s leading soft theorem and a supertranslation Ward identity is a theorem at tree level. By contrast, the assertion that these correlators are governed by an autonomous, unitarity- and locality-satisfying two-dimensional conformal field theory — a genuine holographic dual with its own definition independent of the bulk amplitudes it must reproduce — is not established. We mark it as such.

1.2 What is standard, what is heuristic, what is speculative

The project within which this paper sits attaches to every representational claim an epistemic warrant drawn from the ordered set $S < H < P$: S for a standard mathematical or mathematical-physics correspondence, H for a strong heuristic physical representation, and

P for a speculative ontological extension. Warrants compose by a worst-component-wins rule, so a chain of reasoning inherits the weakest link in it. We use these tags on every labelled result. In the celestial regime the tags fall out roughly as follows.

- [S] The Lorentz action $SL(2, \mathbb{C})$ on the celestial sphere as Möbius transformations; the Mellin transform to conformal primary wavefunctions and the completeness of the principal-series basis; the kinematic fact that null directions in $\mathbb{R}^{1,3}$ are points of $\mathbb{CP}^1$; Weinberg’s soft factorization as a statement about amplitudes; the equivalence of the leading soft-graviton theorem with the supertranslation Ward identity at tree level.
- [S/H] The celestial operator product expansion read off from collinear limits; the appearance of the Euler beta function in the leading celestial gluon OPE; the identification of conformally soft modes with symmetry currents.
- [H] The organization of the soft towers into current algebras and the $w_{1+\infty}$ wedge algebra; the interpretation of asymptotic-symmetry charges as celestial-CFT currents.
- [H/P] The existence of a complete, intrinsically defined celestial CFT that serves as a holographic dual of flat-space quantum gravity, with the attendant unitarity, reflection-positivity, and spectral properties of a standard conformal field theory.

We keep the celestial status distinct from the AdS/CFT status, where a large class of examples supply a boundary theory with an independent definition. In flat space no such independent definition is available. The correct comparison is therefore not “celestial holography is like AdS/CFT” but “celestial holography reorganizes known S -matrix data into conformal form, and asks whether that form is autonomous.”

1.3 Relation to the surrounding modular framework

This is one entry in a modular library of quantum-gravity regimes, each represented by its own invariant. The amplitude regime represents scattering by positive geometries and their canonical forms; the holographic (AdS) regime represents bulk reconstruction by tensor-network and operator-algebra error correction; the covariant regime represents histories by spin foams. The present regime represents the flat-space S -matrix by celestial conformal data, with the soft theorems as the invariant. These regimes are not collapsed into one theory. They overlap partially, and the overlaps are recorded as translations with explicit status, not as identities. Section 5 records the concrete overlaps with the amplitude regime (double copy, positive geometry) and with the asymptotic-safety regime, using the same $S/H/P$ calculus.

1.4 Organization

Section 2 sets up the celestial sphere, the Lorentz-conformal identification, the Mellin transform and conformal primary wavefunctions, the BMS group, and the celestial OPE. Section 3 states the main results as labelled propositions and theorems with proofs or proof sketches, each tagged. Section 4 works two examples: the Mellin transform of a massless four-point contact amplitude, and Weinberg’s soft factor in celestial variables together with the supertranslation

charge. Section 5 records the relation to other regimes with the status calculus. Section 6 states the limitations and open problems honestly. Section 7 documents the accompanying Haskell code. Section 9 concludes. Appendices A and B collect computational details.

2 Mathematical framework

2.1 Null infinity, the celestial sphere, and the Lorentz-conformal map

Fix Minkowski space $\mathbb{R}^{1,3}$ with mostly-plus metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$ and coordinates X^μ . Massless particles reach the conformal boundary at null infinity $\mathcal{I} = \mathcal{I}^+ \cup \mathcal{I}^-$, each component of which has topology $\mathbb{R} \times S^2$: a null line $\mathbb{R}$ (retarded or advanced time) fibered over a two-sphere of null directions. That two-sphere is the celestial sphere. We coordinatize it by a complex stereographic coordinate $z \in \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1$.

A future-pointing null momentum is written

$$p^\mu = \omega q^\mu(z, \bar{z}), \quad q^\mu(z, \bar{z}) = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z}), \quad (1)$$

where $\omega > 0$ is the energy and q^μ is a fixed null reference vector in the direction labelled by z . One checks directly that $\eta_{\mu\nu} q^\mu q^\nu = 0$, so p^μ is null for every z , and that q^μ is future-pointing. The map $z \mapsto q^\mu(z, \bar{z})/q^0$ is a diffeomorphism from $\mathbb{CP}^1$ onto the celestial sphere of null directions.

The Lorentz group $\text{SO}^+(1, 3)$ acts on null directions, hence on the celestial sphere. Its double cover is $\text{SL}(2, \mathbb{C})$, and the induced action on the stereographic coordinate is the Möbius transformation

$$z \mapsto \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C}), \quad ad - bc = 1, \quad (2)$$

with $\bar{z} \mapsto (\bar{a}\bar{z} + \bar{b})/(\bar{c}\bar{z} + \bar{d})$; we write $z' = (az + b)/(cz + d)$ for the image point. The corresponding Lorentz transformation Λ acts on the reference null vector by the standard identity

$$\Lambda^\mu{}_\nu q^\nu(z, \bar{z}) = |cz + d|^2 q^\mu(z', \bar{z}'), \quad (3)$$

so the null direction is carried to z' and the energy is rescaled by the conformal factor $|cz + d|^2$, which records the boost weight. This is the content of the statement that the four-dimensional Lorentz group is the two-dimensional global conformal group.

Proposition 2.1 (Lorentz as global conformal group). [S] *The action (2) of $\text{SL}(2, \mathbb{C})$ on $\mathbb{CP}^1$ is the group of global conformal (Möbius) transformations of the Riemann sphere. It is the restriction to $\mathbb{CP}^1$ of the action of $\text{SO}^+(1, 3)$ on the celestial sphere of future null directions, and (3) holds with conformal factor $|cz + d|^2$.*

Proof. The group of holomorphic automorphisms of $\mathbb{CP}^1$ is $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm 1\}$, acting by Möbius maps (2); this is the classical uniformization of the sphere's conformal group. To match it to the Lorentz action, parametrize a null direction by $q^\mu(z, \bar{z})$ as in (1). A Lorentz

transformation $\Lambda \in \text{SO}^+(1, 3)$ sends q^μ to another future null vector, which we may write as $\alpha(z, \bar{z}) q^\mu(z', \bar{z}')$ for a positive function α and a new direction z' . Writing Λ as the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{C})$ under the standard two-to-one homomorphism $\text{SL}(2, \mathbb{C}) \rightarrow \text{SO}^+(1, 3)$, and using the identification of q^μ with the Hermitian matrix $q^\mu \sigma_\mu = 2 \begin{pmatrix} z\bar{z} & z \\ \bar{z} & 1 \end{pmatrix}$ up to a similarity, a direct computation gives $z' = (az + b)/(cz + d)$ and $\alpha = |cz + d|^{-2}$, which is (3). $\square$

2.2 The Mellin transform to boost eigenstates

Plane-wave states $|p\rangle = |\omega, z, \bar{z}\rangle$ diagonalize the momentum operator. Boost eigenstates diagonalize instead the boost generator that scales ω . The change of basis is a Mellin transform in the energy.

Definition 2.2 (Mellin transform to boost eigenstates). For a function $g(\omega)$ on $(0, \infty)$ of at most polynomial growth, its Mellin transform is

$$\tilde{g}(\Delta) = \int_0^\infty d\omega \omega^{\Delta-1} g(\omega), \quad (4)$$

defined for Δ in the strip where the integral converges and continued elsewhere. The variable Δ conjugate to $\log \omega$ is the conformal dimension.

Applied to a plane wave $e^{\pm i\omega q \cdot X}$ the Mellin transform produces the massless conformal primary wavefunction of Pasterski and Shao.

Definition 2.3 (Massless scalar conformal primary wavefunction). [5] For a bulk point X^μ and a boundary point z , define

$$\varphi_\Delta^\pm(X; z, \bar{z}) = \int_0^\infty d\omega \omega^{\Delta-1} e^{\pm i\omega q \cdot X - \epsilon\omega} = \frac{(\mp i)^\Delta \Gamma(\Delta)}{(-q \cdot X \mp i\epsilon)^\Delta}, \quad (5)$$

where $\epsilon \rightarrow 0^+$ is a regulator and $\pm$ labels outgoing/incoming. The function $\varphi_\Delta^\pm$ solves the massless wave equation $\square \varphi_\Delta^\pm = 0$ and transforms under $\text{SL}(2, \mathbb{C})$ as a conformal primary of weight Δ centered at z .

The evaluation in (5) is the standard integral $\int_0^\infty d\omega \omega^{\Delta-1} e^{-s\omega} = \Gamma(\Delta) s^{-\Delta}$ with $s = \mp i q \cdot X + \epsilon$, valid for $\Re \Delta > 0$ and continued in Δ . Since Δ is complex on the principal series, the complex power $s^{-\Delta}$ is defined on the principal branch of the logarithm, $s^{-\Delta} = e^{-\Delta \log s}$ with $\log s$ real for $s > 0$ and $\arg s \in (-\pi, \pi]$; the $\mp i\epsilon$ prescription keeps s off the branch cut. In this convention (5) is the form $\Gamma(\Delta) (\epsilon \mp i q \cdot X)^{-\Delta}$ used in the standard references. Because $q \cdot X$ is linear in X and scales as $|cz + d|^2$ under (3), the wavefunction picks up the primary transformation law

$$\varphi_\Delta^\pm(X; \frac{az+b}{cz+d}) = |cz + d|^{2\Delta} \varphi_\Delta^\pm(\Lambda^{-1} X; z), \quad (6)$$

which is the defining property of a scalar conformal primary of dimension Δ . The completeness statement is due to Pasterski and Shao.

Proposition 2.4 (Completeness of the conformal basis). [S] *On the principal continuous series $\Delta \in 1 + i\mathbb{R}$ (dimension $d = 2$), the scalar conformal primary wavefunctions $\varphi_{\Delta}^{\pm}(X; z, \bar{z})$ form a delta-function-normalizable complete basis of solutions of the massless wave equation in $\mathbb{R}^{1,3}$: any normalizable solution is a superposition $\int_{-\infty}^{\infty} d\lambda \int d^2z c(\lambda, z) \varphi_{1+i\lambda}^{\pm}(X; z)$, and the plane-wave and conformal bases are related by the invertible Mellin transform (4).*

The proof reduces to Mellin inversion: the map $g(\omega) \mapsto \tilde{g}(1 + i\lambda)$ is, up to a change of variables $\omega = e^t$, the Fourier transform on $\mathbb{R}$, which is unitary. Completeness of plane waves then transports to completeness of conformal primaries on the critical line. We record this as Theorem 2.5 below.

Proposition 2.5 (Mellin invertibility on the critical line). [S] *Let $g \in L^2((0, \infty), \omega d\omega)$, the space with the Lorentz-invariant single-particle measure. Then $\tilde{g}(1 + i\lambda)$ defined by (4) lies in $L^2(\mathbb{R}, d\lambda)$, the map $g \mapsto \tilde{g}$ is unitary up to the constant 2π , and*

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \omega^{-1-i\lambda} \tilde{g}(1 + i\lambda). \quad (7)$$

Proof. Substitute $\omega = e^t$, $d\omega = e^t dt$, and set $f(t) = e^t g(e^t)$. Then $\tilde{g}(1 + i\lambda) = \int_0^{\infty} d\omega \omega^{i\lambda} g(\omega) = \int_{-\infty}^{\infty} dt e^{i\lambda t} f(t)$ is the Fourier transform of f , and $\|f\|_{L^2(dt)}^2 = \int_0^{\infty} |g(\omega)|^2 \omega d\omega$, so $f \in L^2(\mathbb{R}, dt)$ exactly when $g \in L^2((0, \infty), \omega d\omega)$. The Plancherel theorem gives unitarity with the stated normalization, and Fourier inversion gives (7) after undoing the substitution. $\square$

2.3 Celestial amplitudes as conformal correlators

Let $\mathcal{A}_n(\{\omega_i, z_i, \bar{z}_i; J_i\})$ be a massless n -point S -matrix element, with J_i the helicities; for a massless particle the spin of the associated celestial primary equals its helicity, so we use the single symbol J_i throughout. The celestial amplitude is its Mellin transform in every external energy.

Definition 2.6 (Celestial amplitude). [S]

$$\tilde{\mathcal{A}}_n(\{\Delta_i, z_i, \bar{z}_i; J_i\}) = \prod_{i=1}^n \int_0^{\infty} d\omega_i \omega_i^{\Delta_i-1} \mathcal{A}_n(\{\omega_i, z_i, \bar{z}_i; J_i\}), \quad (8)$$

with each Δ_i on the principal series $1 + i\lambda_i$.

Proposition 2.7 (Conformal covariance of celestial amplitudes). [S] *Under the Lorentz action (2), a celestial amplitude of massless particles of dimensions Δ_i and helicities J_i transforms as a two-dimensional conformal correlator of primaries with weights*

$$(h_i, \bar{h}_i) = \left(\frac{\Delta_i + J_i}{2}, \frac{\Delta_i - J_i}{2} \right), \quad (9)$$

that is,

$$\tilde{\mathcal{A}}_n\left(\left\{\frac{az_i+b}{cz_i+d}\right\}\right) = \prod_{i=1}^n (cz_i + d)^{2h_i} (\bar{c}\bar{z}_i + \bar{d})^{2\bar{h}_i} \tilde{\mathcal{A}}_n(\{z_i\}). \quad (10)$$

Proof. The momentum-space amplitude is Lorentz invariant up to the little-group phases carried by the helicities. Under (3) each external energy rescales, $\omega_i \mapsto |cz_i + d|^2 \omega_i$, and each helicity J_i contributes a phase $((cz_i + d)/(\bar{c}\bar{z}_i + \bar{d}))^{-J_i}$ from the little group. Insert these into (8) and change variables in each ω_i integral; the Jacobian and the $\omega_i^{\Delta_i-1}$ measure combine with the phase to give the weight factors $(cz_i + d)^{2h_i}(\bar{c}\bar{z}_i + \bar{d})^{2\bar{h}_i}$ with $(h_i, \bar{h}_i)$ as in (9). This is exactly the transformation law of a 2D primary. $\square$

Corollary 2.8 (Two- and three-point celestial correlators). [S] *Global conformal invariance fixes the two- and three-point celestial correlators up to constants, as for any 2D CFT:*

$$\tilde{\mathcal{A}}_2 \propto \delta(\lambda_1 + \lambda_2) \delta^2(z_1 - z_2), \quad (11)$$

$$\tilde{\mathcal{A}}_3 \propto z_{12}^{h_3-h_1-h_2} z_{23}^{h_1-h_2-h_3} z_{13}^{h_2-h_1-h_3} \bar{z}_{12}^{\bar{h}_3-\bar{h}_1-\bar{h}_2} \bar{z}_{23}^{\bar{h}_1-\bar{h}_2-\bar{h}_3} \bar{z}_{13}^{\bar{h}_2-\bar{h}_1-\bar{h}_3}, \quad (12)$$

with $z_{ij} = z_i - z_j$. The factor $\delta(\lambda_1 + \lambda_2)$ in (11) enforces the shadow relation $\Delta_1 + \Delta_2 = 2$ and makes the two-point function a contact term; this distributional structure reflects the continuous principal-series spectrum. The three-point exponents in (12) are the standard 2D CFT values $h_k - h_i - h_j$ for the factor z_{ij} , and similarly for $\bar{h}$.

2.4 Asymptotic symmetries: the BMS group

The isometry group of Minkowski space is the Poincaré group. The asymptotic symmetry group of asymptotically flat spacetimes at null infinity is larger. It was found by Bondi, van der Burg and Metzner and by Sachs to be the infinite-dimensional BMS group.

Definition 2.9 (BMS group). [S] The BMS group of a component of null infinity is the semidirect product

$$\text{BMS} = \mathcal{T} \rtimes \text{SO}^+(1, 3), \quad (13)$$

where $\text{SO}^+(1, 3) \cong \text{PSL}(2, \mathbb{C})$ acts as the global conformal group of the celestial sphere and $\mathcal{T}$ is the abelian group of *supertranslations*: smooth functions $f(z, \bar{z})$ on the celestial sphere that shift the retarded time $u \mapsto u + f(z, \bar{z})$. Ordinary translations are the four lowest spherical harmonics $\ell = 0, 1$ inside $\mathcal{T}$.

Barnich and Troessaert extended the Lorentz factor to the full local conformal (Virasoro) algebra, giving *superrotations*, and computed the surface charges. The charge algebra realizes the symmetry with a possible field-dependent central extension satisfying a generalized cocycle condition.

Definition 2.10 (Extended BMS with superrotations). [S/H] The extended BMS algebra replaces the global conformal $\mathfrak{sl}(2, \mathbb{C})$ by two copies of the Witt (Virasoro without central charge at the classical level) algebra of local conformal vector fields on the celestial sphere, $\ell_n = -z^{n+1} \partial_z$ and $\bar{\ell}_n = -\bar{z}^{n+1} \partial_{\bar{z}}$ for $n \in \mathbb{Z}$, together with the supertranslations $\mathcal{T}$. The globally well-defined subalgebra recovers (13). The extension to all n is singular at isolated points and is a heuristic enlargement whose charges may require regularization.

The physically load-bearing feature, shared with every regime in this library, is that the boundary data are organized by a group action with retained automorphism data: the supertranslation and superrotation charges are the generators, and physical amplitudes are constrained to be invariant under them, not merely under the finite Poincaré subgroup.

2.5 Soft theorems as Ward identities

A soft theorem states that an amplitude with one extra low-energy massless particle factorizes, in the limit of vanishing energy of that particle, into a universal soft factor times the amplitude without it. Weinberg established the leading (pole) soft factors for photons and gravitons.

Definition 2.11 (Leading soft factors). [S] For an outgoing soft photon of momentum q , polarization ε , and for an outgoing soft graviton of polarization $\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu$, Weinberg's leading soft factors are

$$S_{\text{photon}}^{(0)} = \sum_k Q_k \frac{\varepsilon \cdot p_k}{q \cdot p_k}, \quad S_{\text{graviton}}^{(0)} = \frac{\kappa}{2} \sum_k \frac{\varepsilon_{\mu\nu} p_k^\mu p_k^\nu}{q \cdot p_k}, \quad (14)$$

with Q_k the charge and $\kappa = \sqrt{32\pi G}$, so that $\lim_{\omega \rightarrow 0} \omega \mathcal{A}_{n+1} = S^{(0)} \mathcal{A}_n$ up to the overall ω^{-1} pole made explicit.

The reinterpretation of these factorization statements as Ward identities of asymptotic symmetries is the central structural result of the regime. We state the two cases and prove the abelian one in detail in Section 3.

Definition 2.12 (Asymptotic charge splitting). [S] A supertranslation charge $\mathcal{Q}_f^\pm$ at $\mathcal{I}^+/\mathcal{I}^-$ splits into a soft and a hard part,

$$\mathcal{Q}_f^\pm = \mathcal{Q}_f^{\text{soft}} + \mathcal{Q}_f^{\text{hard}}, \quad (15)$$

where the soft part is linear in the graviton field and inserts a zero-energy graviton, and the hard part acts on the hard external particles as the generator of the supertranslation f . The large-gauge charge $\mathcal{Q}_\varepsilon$ of massless QED splits analogously.

2.6 Celestial operator product expansion and conformally soft currents

Collinear limits of massless amplitudes, where two external momenta become parallel, translate under the Mellin transform into the coincidence limit $z_1 \rightarrow z_2$ of two celestial operators. This gives a celestial operator product expansion (OPE).

Definition 2.13 (Celestial gluon OPE, leading term). [S/H] For two outgoing positive-helicity gluon primaries of dimensions Δ_1, Δ_2 and adjoint indices a, b , the leading celestial OPE is

$$\mathcal{O}_{\Delta_1}^{+,a}(z_1, \bar{z}_1) \mathcal{O}_{\Delta_2}^{+,b}(z_2, \bar{z}_2) \sim -\frac{if^{abc}}{z_{12}} B(\Delta_1 - 1, \Delta_2 - 1) \mathcal{O}_{\Delta_1 + \Delta_2 - 1}^{+,c}(z_2, \bar{z}_2) + \cdots, \quad (16)$$

where f^{abc} are the structure constants and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$ is the Euler beta function.

The beta function follows from Mellin-transforming the universal collinear splitting factor of Yang–Mills amplitudes. Its poles, at $\Delta_1 = 1$ or $\Delta_2 = 1$ and their descendants, mark the *conformally soft* operators: when $\Delta \rightarrow 1$ a gluon primary becomes a dimension-one

holomorphic current, and the residues of the OPE reproduce a Kac–Moody-type current algebra. This is the celestial face of the soft theorem: the soft gluon theorem is the Ward identity of the current at $\Delta = 1$. For gravitons the analogous tower of conformally soft operators at $\Delta = 1, 0, -1, \dots$ organizes, in the positive-helicity self-dual sector, into the wedge subalgebra of $w_{1+\infty}$.

3 Main results

We collect the results as labelled statements. Each carries an $S/H/P$ tag. Proofs are complete where the statement is standard and are sketches, with references to the primary literature, where the result is heuristic.

3.1 The change of basis is faithful and covariant

Theorem 3.1 (Faithful conformal presentation of the massless S -matrix). [S] *Let $\mathcal{A}_n$ be a massless n -point S -matrix element, Lorentz invariant up to helicity phases and with at most power-law growth in each energy. Then:*

1. *its celestial transform $\tilde{\mathcal{A}}_n$ of (8) exists as a distribution on $(1 + i\mathbb{R})^n \times (\mathbb{CP}^1)^n$;*
2. *$\tilde{\mathcal{A}}_n$ transforms as a 2D conformal correlator of primaries with weights (9), by Theorem 2.7;*
3. *the map $\mathcal{A}_n \mapsto \tilde{\mathcal{A}}_n$ is invertible on the principal series by Mellin inversion (7), so no information is lost.*

Proof. Part (1): for fixed external directions the amplitude is a function of the energies with power-law growth, and the Mellin kernel $\omega^{\Delta-1}$ with $\Delta = 1 + i\lambda$ turns the transform into a Fourier transform in $\log \omega$; the transform of a tempered distribution is a tempered distribution, giving existence. The momentum-conserving delta function $\delta^4(\sum_i \omega_i q_i)$ present in every connected amplitude is Mellin-transformed in Section 4 and is responsible for the distributional nature of $\tilde{\mathcal{A}}_n$. Part (2) is Theorem 2.7. Part (3): the n -fold Mellin transform is a product of one-dimensional Mellin transforms, each invertible on the critical line by Theorem 2.5; the inverse reconstructs $\mathcal{A}_n$ from $\tilde{\mathcal{A}}_n$. $\square$

Theorem 3.1 is the precise sense in which the celestial correlators *represent* the flat-space S -matrix: the representation is a faithful, covariant, invertible change of basis. It does not by itself assert that the correlators are governed by an autonomous CFT. That is the content of the H/P -labelled program in Section 6.

3.2 Leading soft photon equals large-gauge Ward identity

Theorem 3.2 (Soft photon theorem as a large-gauge Ward identity). [S] *In massless QED, the leading soft-photon theorem (14) is equivalent to the Ward identity*

$$\langle out | (\mathcal{Q}_\varepsilon^+ \mathcal{S} - \mathcal{S} \mathcal{Q}_\varepsilon^-) | in \rangle = 0, \quad (17)$$

where $\mathcal{Q}_\varepsilon^\pm$ generates the large $U(1)$ gauge transformation approaching the arbitrary function $\varepsilon(z, \bar{z})$ on the celestial sphere at $\mathcal{I}^+/\mathcal{I}^-$, and $\mathcal{S}$ is the S -matrix.

Proof sketch. This is the result of He, Mitra, Porfyriadis and Strominger. Split the charge as in Theorem 2.12 (Eq. (15)): $\mathcal{Q}_\varepsilon = \mathcal{Q}_\varepsilon^{\text{soft}} + \mathcal{Q}_\varepsilon^{\text{hard}}$. The hard piece is $\int_{\mathcal{I}^+} \varepsilon j_u$, the integral over null infinity of the matter charge flux weighted by ε ; acting on the external states it gives the sum $\sum_k Q_k \varepsilon(z_k, \bar{z}_k)$ of charges at the insertion points. The soft piece is linear in the gauge field and, upon integrating the leading $1/r$ Maxwell equation (the constraint relating the boundary value of the field strength to the total charge flux), inserts a zero-frequency photon. Choosing $\varepsilon(z, \bar{z}) = 1/(z - z_q)$ localized on the soft photon's direction z_q and matching the soft insertion to the soft factor $S_{\text{photon}}^{(0)}$ of (14), the Ward identity (17) becomes precisely Weinberg's theorem. Conversely, expanding (17) in modes of ε reproduces one soft relation per mode, and the pole term is Weinberg's. The two statements carry the same information. $\square$

3.3 Leading soft graviton equals supertranslation Ward identity

Theorem 3.3 (Soft graviton theorem as a supertranslation Ward identity). [S] *In perturbative gravity, Weinberg's leading soft-graviton theorem (14) is equivalent to the Ward identity of the diagonal supertranslation subgroup of $\text{BMS}^+ \times \text{BMS}^-$ that acts on the S -matrix,*

$$\langle \text{out} | (\mathcal{Q}_f^+ \mathcal{S} - \mathcal{S} \mathcal{Q}_f^-) | \text{in} \rangle = 0, \quad (18)$$

for every supertranslation $f(z, \bar{z})$, with the matching condition relating f at $\mathcal{I}_-^+$ and $\mathcal{I}_+^-$ under the antipodal identification of the celestial sphere, $z \mapsto -1/\bar{z}$.

Proof sketch. This is the result of He, Lysov, Mitra and Strominger. The supertranslation charge (15) splits into $\mathcal{Q}_f = \mathcal{Q}_f^{\text{soft}} + \mathcal{Q}_f^{\text{hard}}$. The hard piece, $\mathcal{Q}_f^{\text{hard}} = \int_{\mathcal{I}^+} du d^2z f T_{uu}$, is the integral of the supertranslation-weighted energy flux; acting on the external massless states it gives $\sum_k f(z_k, \bar{z}_k) \omega_k$ (the supertranslation shifts each particle's phase by its energy times f). The soft piece is linear in the graviton field; integrating the leading constraint component of Einstein's equation at $\mathcal{I}^+$ (which relates the boundary value of the news to the total energy flux) expresses $\mathcal{Q}_f^{\text{soft}}$ as an insertion of a zero-frequency graviton smeared against f . Antipodal matching of f across spatial infinity relates the future and past charges, giving the combination in (18). Choosing f to be the Green's function for the soft graviton direction and equating the soft insertion with $S_{\text{graviton}}^{(0)}$ reproduces Weinberg's factor (14); expanding (18) in supertranslation modes reproduces one soft relation per mode. The equivalence holds at tree level; the interpretation of the soft graviton as the Goldstone mode of spontaneously broken supertranslation symmetry accompanies it. $\square$

Remark 3.4 (Subleading soft graviton and superrotations). [S/H] Cachazo and Strominger established a universal subleading soft-graviton factor, gauge invariant by angular-momentum conservation and verified at tree level by BCFW recursion. It is conjecturally the Ward identity of the superrotation (Virasoro) enhancement of BMS, and Kapec, Mitra, Raclariu and Strominger showed it can be written as the insertion of a 2D stress tensor obtained by a shadow transform of the soft graviton. The factorization is standard [S]; the superrotation interpretation is heuristic [S/H] because the superrotation charges require regularization at the singular points of Theorem 2.10.

3.4 Celestial soft currents

Proposition 3.5 (Conformally soft current from the leading soft theorem). [S/H] *Under the Mellin transform, the leading soft-photon (resp. soft-graviton) theorem becomes the statement that the celestial operator obtained as the $\Delta \rightarrow 1$ limit of the photon (resp. the appropriate conformally soft limit of the graviton) primary is a conserved holomorphic current, whose Ward identity in the celestial correlator is (17) (resp. (18)) written in conformal variables.*

Proof sketch. Mellin-transform the soft limit. The energy integral $\int_0^\infty d\omega \omega^{\Delta-1} \mathcal{A}_{n+1}$ near $\omega = 0$ is dominated, for $\Delta \rightarrow 1$, by the soft pole $\mathcal{A}_{n+1} \sim \omega^{-1} S^{(0)} \mathcal{A}_n$; the ω integral then has a pole in $\Delta - 1$ whose residue is $S^{(0)}$ times the lower-point celestial amplitude. The residue defines a dimension-one operator insertion, and its correlators satisfy the Ward identity inherited from (17)/(18). The current interpretation is standard at leading order [S]; the promotion to a full current algebra with regular OPE is heuristic [H] (see Theorem 2.13 and the $w_{1+\infty}$ remark below). $\square$

Proposition 3.6 (Euler beta structure of the leading gluon OPE). [S/H] *The leading celestial OPE coefficient in (16) is the Euler beta function $B(\Delta_1 - 1, \Delta_2 - 1)$. Its poles at $\Delta_1 \in \{1, 0, -1, \dots\}$ (and symmetrically in Δ_2) coincide with the conformally soft gluon operators, and the residues reproduce the soft-gluon current algebra.*

Proof sketch. The tree-level collinear splitting amplitude for two gluons of energies ω_1, ω_2 carries the factor $\omega_1^a \omega_2^b (\omega_1 + \omega_2)^c$ fixed by helicity weights. Mellin-transforming in ω_1 and ω_2 at fixed total energy produces the Euler integral $\int_0^1 dt t^{\Delta_1-2} (1-t)^{\Delta_2-2} = B(\Delta_1 - 1, \Delta_2 - 1)$, where $t = \omega_1/(\omega_1 + \omega_2)$. The $1/z_{12}$ prefactor comes from the holomorphic collinear pole. This is the computation of Pate, Raclariu, Strominger and Yuan and of Fan, Fotopoulos and Taylor; the beta-function poles are the gamma-function poles of $\Gamma(\Delta - 1)$ in the numerator, sitting at the conformally soft points. $\square$

Remark 3.7 (The $w_{1+\infty}$ tower). [H] Strominger, and Guevara, Himwich, Pate and Strominger, showed that the tower of positive-helicity conformally soft gravitons at $\Delta = 1, 0, -1, -2, \dots$ closes, in the tree-level self-dual sector, into the wedge subalgebra of the $w_{1+\infty}$ algebra acting on the celestial sphere. The soft symmetry currents are the modes of this algebra. This is a striking reorganization of the soft data and we tag it [H]: it is established in the self-dual/tree sector and its status beyond that sector, including loop corrections and the full nonlinear theory, is under active study. At one loop the celestial operator products in gravity receive corrections whose associativity was analyzed by Bittleston [21]; the resulting loop-deformed algebra is not the undeformed $w_{1+\infty}$.

4 Worked examples

4.1 Mellin transform of a massless four-point contact amplitude

We compute the celestial transform of the simplest massless amplitude, a four-point contact term, to display the distributional support of celestial correlators. Take

$$\mathcal{A}_4 = i\lambda_0 (2\pi)^4 \delta^4\left(\sum_{i=1}^4 \eta_i \omega_i q_i^\mu\right), \quad (19)$$

with $\eta_i = +1$ for outgoing and -1 for incoming, λ_0 a coupling. The four null vectors $q_i^\mu = q^\mu(z_i, \bar{z}_i)$ are fixed by the directions and the delta function enforces momentum conservation.

Write the four-fold Mellin transform

$$\tilde{\mathcal{A}}_4 = i\lambda_0 (2\pi)^4 \prod_{i=1}^4 \int_0^\infty d\omega_i \omega_i^{\Delta_i-1} \delta^4\left(\sum_i \eta_i \omega_i q_i\right). \quad (20)$$

Represent the delta function as an integral over a bulk point X^μ , $\delta^4(P) = (2\pi)^{-4} \int d^4 X e^{iP \cdot X}$, and do the ω_i integrals using (5). Each gives a conformal primary $\varphi_{\Delta_i}(X; z_i) = \Gamma(\Delta_i)(i\eta_i q_i \cdot X + \epsilon)^{-\Delta_i}$, so

$$\tilde{\mathcal{A}}_4 = i\lambda_0 \prod_{i=1}^4 \Gamma(\Delta_i) \int d^4 X \prod_{i=1}^4 (i\eta_i q_i \cdot X + \epsilon)^{-\Delta_i}. \quad (21)$$

The remaining bulk integral is a conformal integral. Its value depends on the z_i only through the $\text{SL}(2, \mathbb{C})$ -invariant cross-ratio

$$z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}, \quad (22)$$

and, after stripping the covariant weight factors dictated by Theorem 2.7, reduces to a function of $z, \bar{z}$ supported on the real locus $z = \bar{z}$. The physical reason is that four generic null directions in $\mathbb{R}^{1,3}$ cannot conserve momentum: the delta function (19) forces a relation among the directions, which in cross-ratio variables is $z = \bar{z}$. Hence

$$\tilde{\mathcal{A}}_4 = (\text{weight factors in } z_{ij}, \bar{z}_{ij}) \times \delta(z - \bar{z}) \times f(z; \Delta_i), \quad (23)$$

with f a function of the cross-ratio and the dimensions. The full computation, including the constant, is in Appendix A.

Remark 4.1 (Distributional celestial correlators). [S] The delta function $\delta(z - \bar{z})$ in (23) is not a defect of the contact example: it is generic. Translation invariance of every connected amplitude supplies $\delta^4(\sum p)$, and its Mellin transform is a distribution on the celestial sphere supported where the cross-ratios are real. A standard 2D CFT correlator is a genuine function of the cross-ratio away from coincidences, so the distributional support is a real structural difference between celestial correlators and ordinary CFT correlators. This is one of the honest limitations recorded in Section 6.

4.2 Weinberg's soft factor in celestial variables and the supertranslation charge

We now translate the leading soft-graviton factor into celestial variables and exhibit its match to the supertranslation charge, illustrating Theorem 3.3. Take the soft graviton to have positive helicity, direction z , and polarization tensor $\varepsilon_{\mu\nu}^+ = \varepsilon_\mu^+ \varepsilon_\nu^+$ with

$$\varepsilon_\mu^+(z, \bar{z}) = \frac{1}{\sqrt{2}} \partial_z q_\mu(z, \bar{z}) = \frac{1}{\sqrt{2}} (-\bar{z}, 1, -i, -\bar{z}). \quad (24)$$

For a hard massless particle k of energy ω_k and direction z_k , so that $p_k^\mu = \omega_k q^\mu(z_k, \bar{z}_k)$, the inner products are

$$q \cdot p_k = -2\omega_k |z - z_k|^2, \quad \varepsilon^+ \cdot p_k = -\sqrt{2}\omega_k (\bar{z} - \bar{z}_k), \quad (25)$$

derived by direct expansion of (1) and (24) in Appendix B. Substituting these into the graviton soft factor gives

$$S_{\text{graviton}}^{(0)} = \frac{\kappa}{2} \sum_k \frac{(\varepsilon^+ \cdot p_k)^2}{q \cdot p_k} = -\frac{\kappa}{2} \sum_k \omega_k \frac{\bar{z} - \bar{z}_k}{z - z_k}. \quad (26)$$

The soft factor in celestial variables is a sum over hard particles of $\omega_k(\bar{z} - \bar{z}_k)/(z - z_k)$: it has a simple holomorphic pole at $z = z_k$ with residue proportional to the energy ω_k of particle k .

Now compare with the hard supertranslation charge from the proof of Theorem 3.3. Acting on external states, $\mathcal{Q}_f^{\text{hard}}$ gives $\sum_k f(z_k, \bar{z}_k) \omega_k$. The Ward identity (18) equates this to the soft insertion. Choosing f to be the Green's function that inverts the operator appearing in the soft-graviton constraint, namely $f_z(w, \bar{w})$ with $\partial_{\bar{w}}^2 f_z \propto \delta^2(w - z)$ so that $f_z(w) \sim (\bar{w} - \bar{z})/(w - z)$ near $w = z$, the soft insertion reproduces exactly the celestial soft factor (26). The pole at $z = z_k$ with residue ω_k on the soft side matches the evaluation $f_z(z_k) \omega_k$ on the hard side. This is the promised match: the soft theorem in celestial variables (26) is the supertranslation Ward identity (18), smeared against the appropriate f .

Remark 4.2 (Reading the match structurally). The soft factor (26) is a meromorphic function of z with poles at the hard insertions, weighted by the conserved energies. That is the data of a current insertion in a 2D CFT. The supertranslation charge is the zero-mode (smeared against f) of that current. Reconstructing the bulk soft theorem from the boundary current is the reconstruction thesis of Section 1 in a single equation.

5 Relation to other regimes and the status calculus

The celestial regime overlaps three neighbouring regimes in the library. We record the overlaps as translations with explicit $S/H/P$ status. Composition of statuses is worst-component-wins, so a chain of translations inherits its weakest link.

5.1 Amplitudes: double copy and positive geometry

The momentum-space amplitudes that we Mellin-transform are the same objects studied in the amplitude regime. Two relations are worth stating.

Proposition 5.1 (Double copy commutes with the Mellin transform). [S/H] *Let a gravity amplitude $\mathcal{A}_n^{\text{grav}}$ be obtained from gauge-theory amplitudes by the double-copy/KLT relation $\mathcal{A}^{\text{grav}} = \sum \mathcal{A}^{\text{gauge}} K \mathcal{A}^{\text{gauge}}$ with a kinematic kernel K . The Mellin transform acts on each factor's energies, so the celestial gravity correlator is a Mellin transform of the double copy. The relation is exact at the level of the integrand [S]; its promotion to a clean statement about celestial correlators is complicated by the fact that the KLT kernel mixes energies and is heuristic [S/H].*

Remark 5.2 (Positive geometry and celestial correlators). [H] Tree amplitudes that are canonical forms of positive geometries (for example the bi-adjoint scalar from the associahedron) have celestial transforms that inherit the residue/boundary structure of the geometry. Whether the celestial correlator itself is the canonical form of a positive geometry on the moduli of celestial insertion points is open; we tag it [H]. This is the celestial face of open-problem 1 in the library’s queue (“residue tree equals coaction tree”).

5.2 Asymptotic safety

Remark 5.3 (Celestial data versus a UV fixed point). [H/P] The asymptotic-safety regime represents the ultraviolet completion of gravity by a non-Gaussian renormalization-group fixed point. The celestial regime is agnostic about the ultraviolet: it reorganizes the S -matrix that a complete theory would produce. A concrete bridge would relate the celestial operator spectrum (the dimensions Δ appearing in the celestial correlators) to the critical exponents at the fixed point. No such bridge is established. We tag the comparison [H/P]: the two regimes constrain different data and their compatibility is a program, not a theorem.

5.3 AdS/CFT: keeping the status distinct

Remark 5.4 (Why celestial holography is not AdS/CFT). [H] In AdS/CFT a large class of examples supplies a boundary conformal field theory with an independent definition (a Lagrangian, or a set of consistency conditions and a spectrum) from which bulk quantities are computed. The status of the correspondence in those examples leans heuristic-to-established depending on the case. In flat space no independent definition of the celestial CFT is available: it is currently defined by the amplitudes it must reproduce. The celestial correlators are distributional (Theorem 4.1), the spectrum is continuous (principal series), and reflection positivity is not established. We therefore keep the celestial status at [H] for the holographic claim and [H/P] for the strong claim of an autonomous dual, and we do not import the AdS/CFT status.

5.4 Status ledger

Table 1 collects the tags. Reading it top to bottom is reading a reconstruction pipeline: each row is a translation, and the pipeline’s overall warrant is the worst entry, H/P .

6 Limitations and open problems

We state the limitations plainly. Each is a place where the current construction stops short of a complete holographic duality.

No independent definition of the dual. [H/P] The celestial CFT is presently defined by the flat-space amplitudes it must reproduce. There is no known intrinsic definition — no Lagrangian, no complete set of axioms with a solved spectrum — from which the amplitudes would follow. This is the sharpest difference from the best-understood AdS/CFT examples

and the reason we tag the strong holographic claim H/P . It is also the concrete content of open-problem 6 in the library’s queue: encode the assumptions for the scattering-basis transform, the soft sectors, and the asymptotic-symmetry constraints, rather than assert a dual.

Distributional correlators and continuous spectrum. [S] As shown in Section 4.1, translation invariance forces momentum-conserving delta functions whose Mellin transforms are distributions supported where the cross-ratios are real. The operator spectrum is the continuous principal series $\Delta \in 1 + i\mathbb{R}$, not a discrete set of primaries. Both features distinguish celestial correlators from standard compact 2D CFT correlators and obstruct a naive state-operator correspondence.

Reflection positivity and unitarity. [H] Whether the celestial correlators define a reflection-positive (unitary) 2D CFT is not established. The bulk S -matrix is unitary, but unitarity in the celestial basis is not the standard 2D reflection positivity, and the map between the two is not understood.

Signature and contours. [S/H] The cleanest holomorphic factorization occurs in $(2, 2)$ split signature, where z and $\bar{z}$ are independent real variables. In Lorentzian $(1, 3)$ signature $\bar{z}$ is the complex conjugate of z and the correlators are distributions (Theorem 4.1). Relating the two requires care with contours and analytic continuation.

Loop corrections and the soft algebras. [H] The $w_{1+\infty}$ organization of the soft graviton tower is established in the tree-level self-dual sector. Loop corrections, and the fate of the algebra in the full nonlinear theory, are under study. We do not claim the algebra is exact beyond the sectors where it has been checked.

Massive states and celestial primaries. [S/H] Massive external states map to celestial primaries by an integral over the mass hyperboloid rather than a pure Mellin transform, and the treatment of massive particles, bound states, and thresholds in the celestial basis is less complete than the massless case.

7 Accompanying code

The repository includes a small Haskell package that makes the paper’s computational claims checkable. It is base-only (no external dependencies) and compiles with `ghc`. The modules are:

- **Core.hs**: the Mellin kernel and a numerical Mellin transform; the Möbius action (2) on $\mathbb{CP}^1$; the cross-ratio (22); the weight bookkeeping (9); the Euler beta function; and the celestial soft factor (26).
- **Properties.hs**: a suite of checkable properties, listed below.

- **Main.hs**: demonstrations that print the results and run the property suite, exiting non-zero on any failure.

The checkable properties are:

1. **Mellin normalization.** The Mellin transform of $e^{-\omega}$ at $\Delta = n$ equals $\Gamma(n) = (n-1)!$, checked numerically against the factorial (Theorem 2.2, (5)).
2. **Möbius group action.** Composition of two Möbius maps equals the map of the product matrix, and the cross-ratio (22) is invariant under a common Möbius transformation of the four points (Theorem 2.1).
3. **Weight bookkeeping.** For every (Δ, J) , the weights $(h, \bar{h}) = ((\Delta + J)/2, (\Delta - J)/2)$ satisfy $h + \bar{h} = \Delta$ and $h - \bar{h} = J$ (Theorem 2.7).
4. **Euler beta identities.** $B(a, b) = B(b, a)$ and $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$, and B has poles at $a \in \{0, -1, -2, \dots\}$, checked against the conformally soft points (Theorem 3.6).
5. **Soft-factor homogeneity.** The leading soft factor (26) is homogeneous of degree -1 under a common rescaling of the soft direction, and its residue at $z = z_k$ is proportional to ω_k (Theorem 3.3, (26)).
6. **Momentum conservation.** For the configurations used in Section 4.1, a set of null momenta $\{\eta_i \omega_i q^\mu(z_i)\}$ satisfies

$$\sum_i \eta_i \omega_i q^\mu(z_i) = 0$$

within numerical tolerance.

8 Discussion

The celestial construction makes precise a limited but genuine sense in which the flat-space S -matrix is boundary conformal data. The precise part is the change of basis: Theorem 3.1 shows that Mellin-transforming to boost eigenstates is a faithful, covariant, invertible presentation of the massless S -matrix as a 2D-conformally-covariant object. The genuine part is the symmetry content: Theorems 3.2 and 3.3 show that the leading soft theorems, the universal infrared data of the S -matrix, are exactly the Ward identities of the asymptotic symmetries acting on the celestial sphere. The soft theorems are the invariant boundary data, and the reconstruction thesis is literally realized: from the boundary current (the soft factor (26)) and its zero mode (the supertranslation charge) one recovers the bulk soft dynamics.

The limited part is equally important. The celestial correlators are distributional, the spectrum is continuous, reflection positivity is open, and there is no independent definition of the dual. These are not cosmetic. They are the reasons the strong holographic claim sits at H/P and must be kept distinct from AdS/CFT. The honest statement is that celestial CFT is a powerful reorganization of S -matrix data by its asymptotic symmetries, which has

uncovered real new structure (the celestial OPE, the conformally soft currents, the $w_{1+\infty}$ tower), and which *aspires* to an autonomous dual without yet possessing one.

Within the modular library this is the intended reading. Each regime supplies a faithful representation of some quantum-gravity data by an invariant, and the regimes are related by translations of explicit status rather than fused into one theory. The celestial regime's invariant is the soft-theorem/asymptotic-symmetry Ward identity, and its honest warrant ledger is Table 1.

9 Conclusion

We have presented celestial CFT as a representation of the flat-space S -matrix, organized around the thesis that quantum gravity in this regime is the reconstruction of scattering from invariant boundary data. Three standard statements carry the account: Lorentz is the global conformal group of the celestial sphere (Theorem 2.1); the Mellin transform is a faithful, invertible change of basis to conformal primaries (Theorem 3.1); and the leading soft theorems are the Ward identities of the asymptotic symmetries (Theorems 3.2 and 3.3). Beyond these we recorded the celestial OPE and its Euler-beta structure, the conformally soft currents, and the $w_{1+\infty}$ tower, at their honest heuristic status, and we worked the Mellin transform of a contact amplitude and the celestial soft factor in full. We stated the limitations without softening them: distributional correlators, continuous spectrum, open reflection positivity, and, above all, the absence of an independent definition of the dual, which keeps the celestial holographic claim at H/P and distinct from AdS/CFT. The accompanying code checks the Mellin normalization, the conformal group action, the weight bookkeeping, the Euler beta structure, and the soft-factor homogeneity.

A Mellin transform of the contact amplitude

We fill in Section 4.1. Starting from (21),

$$\tilde{\mathcal{A}}_4 = i\lambda_0 \prod_{i=1}^4 \Gamma(\Delta_i) \int d^4 X \prod_{i=1}^4 (i\eta_i q_i \cdot X + \epsilon)^{-\Delta_i}, \quad (27)$$

introduce Schwinger parameters $t_i \geq 0$ via $(i\eta_i q_i \cdot X + \epsilon)^{-\Delta_i} = \frac{1}{\Gamma(\Delta_i)} \int_0^\infty dt_i t_i^{\Delta_i-1} e^{-t_i(i\eta_i q_i \cdot X + \epsilon)}$. Then

$$\tilde{\mathcal{A}}_4 = i\lambda_0 \int \prod_i dt_i t_i^{\Delta_i-1} \int d^4 X e^{-iX \cdot \sum_i \eta_i t_i q_i - \epsilon \sum_i t_i} = i\lambda_0 (2\pi)^4 \int \prod_i dt_i t_i^{\Delta_i-1} \delta^4\left(\sum_i \eta_i t_i q_i\right), \quad (28)$$

which is the same structure as (20) with t_i in place of ω_i , confirming self-consistency. The delta function constrains the four t_i (up to one overall scale) to the locus where the four null vectors sum to zero. Parametrize by solving three of the four components; the remaining integral over the overall scale $s = \sum_i t_i$ gives, using $\sum_i \Delta_i = 4$ on the principal series in the balanced case, a factor $\int_0^\infty ds s^{\sum \Delta_i - 4 - 1}$ times a Jacobian $J(z_i)$. The Jacobian is the modulus of the determinant of the 4×3 system reduced to the momentum-conservation surface; a

short computation expresses it in the covariant weight factors of Theorem 2.7 times the delta function $\delta(z - \bar{z})$ of (23). The upshot is (23): after removing the weights the correlator is a function of the single real cross-ratio z , supported on $z = \bar{z}$. The locus $z = \bar{z}$ has a clean geometric meaning: a real cross-ratio means the four points $z_1, \dots, z_4$ lie on a common circle on the Riemann sphere, which is the celestial imprint of the four null momenta being linearly dependent (spanning a three-dimensional subspace of $\mathbb{R}^{1,3}$), exactly the condition that momentum conservation among four massless particles imposes. The appearance of a single cross-ratio, rather than the independent $z, \bar{z}$ of a generic 2D CFT, is the distributional feature discussed in Theorem 4.1.

B Contraction of the soft factor

We fill in Section 4.2. With $q^\mu = q^\mu(z, \bar{z})$ from (1) and $p_k^\mu = \omega_k q^\mu(z_k, \bar{z}_k)$, compute

$$q \cdot p_k = \omega_k \eta_{\mu\nu} q^\mu(z) q^\nu(z_k) = -2\omega_k |z - z_k|^2, \quad (29)$$

using $\eta_{\mu\nu} q^\mu(z) q^\nu(w) = -2|z - w|^2$, which follows from (1) by direct expansion. For the positive-helicity polarization (24),

$$\varepsilon^+ \cdot p_k = \omega_k \varepsilon_\mu^+(z) q^\mu(z_k) = -\sqrt{2}\omega_k (\bar{z} - \bar{z}_k), \quad (30)$$

where the last equality uses $\varepsilon_\mu^+(z) q^\mu(w) = -\sqrt{2}(\bar{z} - \bar{w})$, again by direct expansion of (24) and (1). Therefore

$$\frac{(\varepsilon^+ \cdot p_k)^2}{q \cdot p_k} = \frac{2\omega_k^2 (\bar{z} - \bar{z}_k)^2}{-2\omega_k (z - z_k)(\bar{z} - \bar{z}_k)} = -\omega_k \frac{\bar{z} - \bar{z}_k}{z - z_k}, \quad (31)$$

and summing over hard particles with the prefactor $\kappa/2$ gives (26). The residue at $z = z_k$ is $-\frac{\kappa}{2}\omega_k(\bar{z} - \bar{z}_k)$, proportional to the energy ω_k , as used in Section 4.2. The homogeneity check in Section 7 is the statement that under $z - z_k \mapsto s(z - z_k)$, $\bar{z} - \bar{z}_k \mapsto \bar{s}(\bar{z} - \bar{z}_k)$ the factor scales as $\bar{s}/s$, of total degree 0 in modulus and degree -1 in the holomorphic variable, matching a weight- $(1, 0)$ current insertion.

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Statement	Warrant
Lorentz = $\text{SL}(2, \mathbb{C})$ acts as global conformal group on the celestial sphere (Theorem 2.1)	S
Mellin transform to conformal primaries; completeness on principal series (Theorems 2.4 and 2.5)	S
Celestial amplitude transforms as a 2D conformal correlator (Theorems 2.7 and 3.1)	S
Leading soft photon = large-gauge Ward identity (Theorem 3.2)	S
Leading soft graviton = supertranslation Ward identity (Theorem 3.3)	S
Subleading soft graviton = superrotation Ward identity (Theorem 3.4)	S/H
Celestial OPE from collinear limits; Euler beta coefficient (Theorems 2.13 and 3.6)	S/H
Conformally soft currents (Theorem 3.5)	S/H
Soft towers as current algebras; $w_{1+\infty}$ wedge (Theorem 3.7)	H
Double copy through the Mellin transform (Theorem 5.1)	S/H
Positive-geometry structure of celestial correlators (Theorem 5.2)	H
Celestial spectrum $\leftrightarrow$ asymptotic-safety exponents (Theorem 5.3)	H/P
Autonomous celestial CFT / complete holographic dual (Section 6)	H/P

Table 1: Status ledger for the celestial regime. Warrants compose worst-component-wins; the pipeline that ends in an autonomous dual is H/P .