Observables as Cohomology: BV–BRST Complexes and Derived Critical Loci as the Gauge Layer of Reconstructed Spacetime

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Abstract

We treat the gauge and constraint layer of quantum gravity not as a nuisance to be removed before quantization but as invariant representation structure whose cohomology reconstructs the physical observable algebra. The technical vehicle is the Batalin–Vilkovisky (BV) and Becchi–Rouet–Stora–Tyutin (BRST) formalism, read through the lens of derived algebraic geometry: the on-shell gauge quotient of a classical field theory is modelled by the derived critical locus $\mathrm{dCrit}(S)$ of the action, a (-1)-shifted symplectic derived scheme whose structure complex is the classical BV complex. Physical observables are the degree-zero cohomology H^0 of the BV differential $Q_{\mathrm{BV}} = (S, -)$. We give a self-contained account of the antibracket, the classical master equation, the Koszul–Tate resolution of the equation-of-motion ideal, the Chevalley–Eilenberg model of gauge invariance, and their assembly into a single bigraded complex. Our results are stated as labelled theorems and each is tagged with the project's S/H/P epistemic calculus (standard / heuristic / speculative): the identification of the antibracket with the PTVV (-1)-shifted Poisson structure is standard mathematics; the identification of H^0 with the physical observable algebra of an interacting quantum theory is heuristic, contingent on the vanishing of a cohomological quantization obstruction; and any extension to a nonperturbative, background-independent observable algebra is speculative. We work three examples in full: a finite abelian gauge system where every cohomology group is computed by hand, a nonabelian Lie-algebra model whose H^0 is the invariant ring, and the reducible case relevant to gravity, where a background Killing vector forces a nontrivial ghost-for-ghost tower dual to the isometry algebra. A companion Haskell development implements a finite BV toy model and checks $s^2 = 0$ and $Q_{\mathrm{BV}}^2 = 0$ by exhaustive evaluation. We close by composing this regime with the other eleven regimes of the modular library through the weakest-link status calculus, and list what remains open, in particular the absence of an explicit ghost-for-ghost computation at Kerr and of a global derived-stack model of the diffeomorphism quotient in four dimensions.

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1 Introduction

1.1 The reconstruction thesis, restricted to the gauge layer

This paper is one module of a deliberately modular program whose governing sentence is:

> Quantum gravity is not primarily the quantization of material objects in spacetime. It is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures.

Twelve regimes carry that program, each responsible for one layer. The present module is the gauge, descent, and observable layer. Its single organizing claim is short:

> Observables are gauge-invariant cohomology classes.

The word "reconstruction" is doing real work here. In the textbook picture one starts with a gauge field, notices that many field configurations describe the same physics, and then quotients or gauge-fixes to remove the redundancy so that quantization can proceed. The redundancy is treated as an obstacle. In the reconstruction picture the redundancy is data. What is physical is not a point in a quotient set but a class in a cohomology theory, and the cohomology retains exactly the information a naive quotient throws away: the stabilizer of a configuration, the relations among gauge generators, the relations among those relations. For gravity this retained information is the isometry (Killing) content of a background, and it is not decoration. It controls degeneracies, conserved charges, and the reducibility structure of the perturbative expansion.

The mathematical statement of "physical content is a class, not a point" is that the correct object is a derived quotient, or equivalently a groupoid or stack, rather than a coarse quotient. At the level of a single action functional S on a space of fields, the correct on-shell gauge-invariant object is the *derived critical locus* $\mathrm{dCrit}(S)$: the derived intersection of the graph of dS with the zero section of the cotangent bundle. Its ring of functions, with a canonical differential, is precisely the classical BV complex. The degree-zero cohomology of that complex is the algebra of on-shell gauge-invariant functionals. This is the technical core of the module, and the rest of the paper unpacks it.

1.2 What is standard, what is heuristic, what is speculative

The program attaches to every dictionary entry a status drawn from an ordered set of three warrants, $\mathsf{S} < \mathsf{H} < \mathsf{P}$: **S** for a standard mathematical or mathematical-physics correspondence, **H** for a strong heuristic physical representation, **P** for a speculative ontological extension. Statuses compose by the worst component, so no chain of reasoning can be advertised at a status better than its weakest link. We take this discipline seriously and tag each result. To preview the honest accounting:

- The identification of the BV antibracket with the (-1)-shifted Poisson bracket on $\mathrm{dCrit}(S)$, the nilpotency $Q_{\mathrm{BV}}^2 = 0$ as the classical master equation, and the computation of H^0 as invariant functions on the critical locus, are all `[S]`: established mathematics.
- The identification of that H^0 with the physical observable algebra of the *interacting quantum* theory is `[S/H]`: it holds when a tower of cohomological obstructions vanishes order by order in $\hbar$ [7, 8], and that vanishing is the rigorous content of "renormalizable and anomaly-free."
- Any claim that this observable algebra extends to a nonperturbative, background-independent theory of quantum gravity is `[H]` at best and, when chained to the more speculative regimes of the library, `[P]`. We do not make such a claim; we bound it.

1.3 Contributions

Three things in this paper are ours rather than a rehearsal of the literature.

First (section 8), a fully explicit finite-dimensional BV model whose every cohomology group is computed by hand and, independently, by an executable Haskell program that checks the nilpotency of the BRST and BV differentials by exhaustive evaluation on a basis. This discharges, for a finite model, the project's formalization item.

Second (section 9), a clean statement in the representation-stack language of the library of the fact that a background with a nontrivial isometry group forces a nontrivial ghost-for-ghost tower whose lowest reducibility degree is dual to the isometry Lie algebra. This ties the classical slice-theorem picture of the gravitational moduli stack to the perturbative-quantum ghost tower, and it is the BV-level shadow of "isotropy is retained, not erased."

Third (section 11), a worked application of the weakest-link status calculus that places this regime among the other eleven, including the sharp computation that BV-BRST composed with an indefinite-causal-order regime lands at speculative status no matter how careful the BV construction is.

1.4 Relation to prior work

The antibracket and antifield formalism are due to Batalin and Vilkovisky [1, 2], with the reducible (linearly dependent generator) case in the second paper. The homological reading, Koszul–Tate resolution and BRST differential, is developed in Henneaux and Teitelboim [3] and, for the local cohomology $H(s|d)$, in Barnich, Brandt and Henneaux [4]. The functional and locally covariant formulation appropriate to field theory on curved spacetime is due to Fredenhagen and Rejzner [10, 11]. The derived-geometric foundation, in which $\mathrm{dCrit}(S)$ carries a canonical (-1)-shifted symplectic structure, is Pantev, Toën, Vaquié and Vezzosi [5]; its symplectic reduction and the comparison with classical BV is Anel and Calaque [6]. The factorization-algebra package, in which the quantization of a classical BV theory is controlled by an obstruction cocycle, is Costello [7] and Costello–Gwilliam [8, 9]. The L_∞ repackaging of the data is Jurčo, Raspollini, Sämann and Wolf [12]. For gravity specifically, the BV-BFV treatment of the Einstein–Hilbert action is Cattaneo and Schiavina [14], building on the boundary axioms of Cattaneo, Mnev and Reshetikhin [13]. Our job is to assemble these into the reconstruction narrative and to tag the seams honestly.

1.5 Outline

Section 2 recalls the representation-stack setting and the status calculus. Section 3 sets up graded geometry and the antibracket. Section 4 builds the BRST complex from ghosts and the Chevalley–Eilenberg differential. Section 5 builds the Koszul–Tate resolution and states the classical master equation. Section 6 gives the derived-critical-locus picture and the (-1)-shifted symplectic structure. Section 7 states the main results on observables as cohomology. Section 8 works the finite abelian and nonabelian examples. Section 9 treats the reducible gravitational case. Section 10 discusses the BV Laplacian, the quantum master equation and the obstruction. Section 11 composes this regime with the other eleven. Section 12 lists limitations and open problems. Section 13 concludes.

2 The representation-stack setting and the status calculus

We record the ambient structure the module inherits, so the paper is self-contained.

2.1 Representation entries and the realization pipeline

A *representation entry* is a tuple $E = (M, P, \tau, \sigma)$: a mathematical object M, a physical target P, a translation datum τ relating them, and an epistemic status σ. A collection of entries forms a *representation prestack*, a pseudofunctor $\mathrm{Rep}_{\mathrm{phys}} \colon \mathrm{Dom}^{\mathrm{op}} \to \mathrm{Gpd}$ from a category of domains to groupoids. It is a *representation stack* for a Grothendieck topology when locally defined, pairwise-compatible entries glue to a global entry uniquely up to coherent isomorphism. The point of insisting on a stack rather than a mere presheaf of sets is that gluing is by equivalence, not equality, so automorphism data (gauge redundancy, stabilizers, isotropy) is retained. Every regime of the library is, in its own way, an instance of the slogan

$$\text{physical content} \;=\; \text{groupoid/stack}, \qquad \mathrm{Aut}_{[X/G]}(x) \;\simeq\; \mathrm{Stab}_G(x). \tag{1}$$

The present module is the instance in which X is a space of fields, G is a gauge group (or, infinitesimally, a gauge Lie algebroid), and the homotopical quotient $[X/G]$ intersected with the shell $\{dS = 0\}$ is the derived critical locus.

2.2 The status calculus

Definition 2.1 (Warrants and status)**.** Let $\mathsf{S} < \mathsf{H} < \mathsf{P}$ be a linearly ordered three-element set of warrants. A *status* is a nonempty tuple of warrants, rendered as one of S, H, P, or a composite such as S/H. The *worst* of a status is the maximum of its components under the order. Composition of statuses is

$$\mathrm{compose}(\sigma_1, \ldots, \sigma_n) \;=\; \max\{\mathrm{worst}(\sigma_1), \ldots, \mathrm{worst}(\sigma_n)\}, \tag{2}$$

the worst component over all inputs.

Proposition 2.2 (Monotonicity and unit)**.** *Composition* (2) *is associative and commutative, is monotone in each argument for the order* $\mathsf{S} < \mathsf{H} < \mathsf{P}$*, and has* S *as a two-sided unit:* $\mathrm{compose}(\mathsf{S}, \sigma) = \sigma$ *for every* σ*.*

Proof. The operation is max on the image of the warrant tuples in the totally ordered set $\{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$. Maximum on a totally ordered set is associative, commutative and monotone; the least element S is its identity. `[S]` $\square$

Theorem 2.2 is the formal reason the framework is modular rather than unified. Chaining regimes never improves the epistemic warrant of the composite beyond the weakest link. We will invoke this repeatedly, and it is checked in the companion code by an exhaustive test over the (finite) status set.

3 Graded geometry and the antibracket

3.1 Graded manifolds of fields

Fix a field $\Bbbk$ of characteristic zero (we take $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{Z}$ depending on whether we track signs numerically). Physical BV data live on a $\mathbb{Z}$-graded manifold; for a self-contained account we present the local, finite-dimensional algebraic model and treat the field-theoretic case as its formal or local-functional completion, following [10, 8].

Definition 3.1 (Graded symmetric algebra of fields)**.** Let $V = \bigoplus_{k\in\mathbb{Z}} V_k$ be a $\mathbb{Z}$-graded $\Bbbk$-vector space, finite dimensional in each degree. The *algebra of functions* on the graded manifold $V[1]^*$-style model is the graded-commutative algebra

$$\mathcal{O}(V) \;=\; \mathrm{Sym}\big(V^*\big) \;=\; \bigoplus_{n\geq 0} \mathrm{Sym}^n\big(V^*\big), \tag{3}$$

where $V^* = \bigoplus_k V_k^*$ with V_k^* in degree $-k$, and the graded symmetric algebra imposes

$$ab \;=\; (-1)^{|a||b|}\, ba \tag{4}$$

for homogeneous a, b of degrees $|a|, |b|$. Elements of even degree commute; elements of odd degree anticommute and square to zero.

We assign to generators several integer gradings that will be used in bookkeeping. The *ghost number* gh is the total $\mathbb{Z}$-grading. It is refined into the *pure ghost number* pgh ≥ 0, carried by ghosts, and the *antifield number* afn ≥ 0, carried by antifields, with

$$\mathrm{gh} \;=\; \mathrm{pgh} - \mathrm{afn}. \tag{5}$$

The Grassmann parity is gh mod 2 for the standard field content; we keep parity and ghost number distinct because for reducible theories they can differ in intermediate constructions.

3.2 The odd symplectic form and the antibracket

The defining feature of BV geometry is an odd Poisson bracket of degree +1, the *antibracket*. Concretely, the field content is doubled: to every generator ϕ^A (fields, ghosts, ghosts-for-ghosts) one adjoins an *antifield* ϕ^*_A of complementary degree,

$$\mathrm{gh}(\phi^*_A) \;=\; -\mathrm{gh}(\phi^A) - 1, \qquad \varepsilon(\phi^*_A) \;=\; \varepsilon(\phi^A) + 1 \pmod 2, \tag{6}$$

where ε is Grassmann parity. The antibracket of two functionals F, G on the doubled space is

$$(F, G) \;=\; \frac{\partial^R F}{\partial \phi^A}\frac{\partial^L G}{\partial \phi^*_A} \;-\; \frac{\partial^R F}{\partial \phi^*_A}\frac{\partial^L G}{\partial \phi^A}, \tag{7}$$

with a sum over A, where ∂^R and ∂^L denote right and left derivatives. The bracket carries ghost number +1 and reverses Grassmann parity.

Proposition 3.2 (Graded symmetry and Jacobi)**.** *The antibracket* (7) *satisfies, for homogeneous F, G, H of ghost numbers written as subscripts where needed,*

$$(F, G) = -(-1)^{(\varepsilon_F+1)(\varepsilon_G+1)}\,(G, F), \tag{8}$$
$$(F, GH) = (F, G)H + (-1)^{(\varepsilon_F+1)\varepsilon_G} G\,(F, H), \tag{9}$$
$$0 = (-1)^{(\varepsilon_F+1)(\varepsilon_H+1)}\big(F, (G, H)\big) + \textit{cyclic}. \tag{10}$$

That is, $(-,-)$ *is an odd Poisson bracket of degree* +1*: a Gerstenhaber bracket on* $\mathcal{O}(\mathcal{F})$*.*

Proof. Each identity is a direct computation from (7) using the graded Leibniz rule for left and right derivatives and the sign conventions (6). The antisymmetry (8) follows because exchanging the two terms of (7) and relabelling produces the stated sign; (9) is the Leibniz rule for the constituent derivatives; (10) is the standard verification that the odd bracket built from a constant odd symplectic form obeys the graded Jacobi identity. Full sign bookkeeping is in [3, Ch. 15, 18] and [10]. [S] □

Remark 3.3 (Geometric origin). The bracket (7) is the Poisson bracket of the constant odd symplectic form $\omega_{\mathrm{BV}} = \sum_A \delta\phi^A \wedge \delta\phi^*_A$ of ghost number -1 on the shifted cotangent bundle $T^*[-1]\mathcal{F}_0$, where $\mathcal{F}_0$ is the space of fields-and-ghosts. In the derived-geometric language of section 6 this odd symplectic form is exactly the (-1)-shifted symplectic structure that PTVV [5] attach to a derived critical locus. [S]

3.3 The BV differential

Definition 3.4 (BV action and differential)**.** A *BV action* is an element $S \in \mathcal{O}(\mathcal{F})$ of ghost number 0 and even parity solving the *classical master equation*

$$(S, S) \;=\; 0. \tag{11}$$

The associated *BV differential* is

$$Q_{\mathrm{BV}} \;=\; (S, -)\colon \mathcal{O}(\mathcal{F}) \to \mathcal{O}(\mathcal{F}), \tag{12}$$

an operator of ghost number +1.

Proposition 3.5 (Nilpotency)**.** *If S satisfies the classical master equation (11), then $Q_{\mathrm{BV}}^2 = 0$.*

Proof. For any F,

$$Q_{\mathrm{BV}}^2 F = \big(S, (S, F)\big). \tag{13}$$

By the graded Jacobi identity (10) applied to (S, S, F) and the fact that S has even total parity (so it is odd for the shifted bracket), one gets

$$\big(S, (S, F)\big) = \tfrac{1}{2}\big((S, S), F\big) = 0 \tag{14}$$

using (11). The factor $\frac{1}{2}$ arises because (S, S) appears with a symmetric coefficient in the Jacobi identity for the odd bracket. [S] □

Theorem 3.5 is the linchpin: the master equation is precisely the integrability condition making Q_{BV} a differential, so that cohomology is defined. The companion Haskell program verifies both the master equation and $Q_{\mathrm{BV}}^2 = 0$ for the finite model of section 8 by exhaustive evaluation, giving an independent machine check of theorem 3.5 in that case.

4 The BRST complex: ghosts and gauge invariance

We now specialize to the geometry of a gauge symmetry and recover the classical BRST complex as one half of the BV complex. Throughout, the gauge symmetry is generated by a Lie algebra (or Lie algebroid) $\mathfrak{g}$ acting on a space X with coordinate ring $\mathcal{O}(X)$.

4.1 The Chevalley–Eilenberg model of invariance

Definition 4.1 (Chevalley–Eilenberg algebra)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra acting on the commutative algebra $\mathcal{O}(X)$ by derivations, $a \mapsto \rho(\xi)a$ for $\xi \in \mathfrak{g}$. Introduce ghosts c^α of ghost number +1 (odd), dual to a basis $\{T_\alpha\}$ of $\mathfrak{g}$ with structure constants $[T_\alpha, T_\beta] = f^\gamma{}_{\alpha\beta} T_\gamma$. The *Chevalley–Eilenberg algebra* is

$$\mathrm{CE}(\mathfrak{g}, \mathcal{O}(X)) \;=\; \mathcal{O}(X) \otimes \Lambda^\bullet \mathfrak{g}^* \;=\; \mathcal{O}(X)[c^1, \ldots, c^{\dim \mathfrak{g}}], \tag{15}$$

graded by pure ghost number $\mathrm{pgh}(c^\alpha) = 1$, with differential

$$s \;=\; c^\alpha \rho(T_\alpha) \;-\; \tfrac{1}{2} f^\gamma{}_{\alpha\beta}\, c^\alpha c^\beta \frac{\partial}{\partial c^\gamma}. \tag{16}$$

Proposition 4.2 ($s^2 = 0$)**.** *The operator* (16) *satisfies* $s^2 = 0$ *if and only if* ρ *is a Lie algebra action and the* ${f^\gamma}_{\alpha\beta}$ *satisfy the Jacobi identity.*

Proof. Compute s^2 on a function $a \in \mathcal{O}(X)$ and on a ghost c^γ separately. On a,

$$s^2 a = c^\alpha c^\beta \big(\rho(T_\beta)\rho(T_\alpha)a\big) - \tfrac{1}{2}{f^\gamma}_{\alpha\beta}c^\alpha c^\beta \rho(T_\gamma)a. \tag{17}$$

Antisymmetrizing the first term in α, β (forced by the odd ghosts) gives $\frac{1}{2}c^\alpha c^\beta[\rho(T_\beta), \rho(T_\alpha)]a = -\frac{1}{2}c^\alpha c^\beta \rho([T_\alpha, T_\beta])a$, which cancels the second term precisely because ρ is a representation, $\rho([T_\alpha, T_\beta]) = [\rho(T_\alpha), \rho(T_\beta)]$. On c^γ, the vanishing of s^2 is the Jacobi identity ${f^\delta}_{\alpha\beta}{f^\epsilon}_{\delta\gamma}+\text{cyclic} = 0$. Both computations are carried out symbolically in the companion code for $\mathfrak{g} = \mathfrak{su}(2)$ and for an abelian $\mathfrak{g}$. [S] $\square$

Theorem 4.3 (BRST cohomology in ghost number zero is the invariants)**.** *For the complex* $(\mathrm{CE}(\mathfrak{g}, \mathcal{O}(X)), s)$*,*

$$H^0(s) \;=\; \mathcal{O}(X)^{\mathfrak{g}} \;=\; \{a \in \mathcal{O}(X) : \rho(\xi)a = 0 \;\forall \xi \in \mathfrak{g}\}, \tag{18}$$

the ring of gauge-invariant functions, and when G *is a compact (equivalently, linearly reductive) group with Lie algebra* $\mathfrak{g}$ *acting locally freely, so that the action on* $\mathcal{O}(X)$ *is completely reducible,* $H^k(s) \cong H^k(\mathfrak{g}) \otimes \mathcal{O}(X/G)$ *computes the Lie algebra cohomology tensored with functions on the honest quotient.*

Proof. In pure ghost number 0 the complex is $\mathcal{O}(X) \xrightarrow{s} \mathcal{O}(X) \otimes \mathfrak{g}^*$ with $sa = c^\alpha \rho(T_\alpha)a$. The kernel is exactly $\{a : \rho(T_\alpha)a = 0 \;\forall \alpha\} = \mathcal{O}(X)^{\mathfrak{g}}$, and there is nothing in negative pure ghost number to bound, so $H^0(s) = \mathcal{O}(X)^{\mathfrak{g}}$. The higher statement is the standard computation of Lie algebra cohomology with coefficients in $\mathcal{O}(X)$; when G is compact (so $\mathcal{O}(X)$ is completely reducible and an equivariant projection onto invariants exists) and acts locally freely, an averaging (Van Est) argument reduces it to $H^\bullet(\mathfrak{g}) \otimes \mathcal{O}(X)^{\mathfrak{g}}$ [3, Ch. 8]. Compactness (or linear reductivity) is essential here: for a non-reductive, unipotent action the coefficient module need not be completely reducible and the tensor-product formula can fail, as section 8.1 illustrates. [S] $\square$

Theorem 4.3 is the cohomological reconstruction of the naive quotient: the invariant functions $\mathcal{O}(X/G)$ are recovered as H^0, but the full cohomology retains the group-theoretic data $H^\bullet(\mathfrak{g})$ that the coarse quotient forgets. This is (1) at the level of functions.

4.2 Field-antifield doubling and the classical BRST charge

To couple the BRST differential to the dynamics one adds antifields and requires the master equation. Write ϕ^i for the original fields with antifields ϕ^*_i, and c^α for ghosts with antifields c^*_α. The minimal BV action for an irreducible gauge theory with action $S_0(\phi)$ invariant under $\delta_\epsilon \phi^i = {R^i}_\alpha(\phi)\epsilon^\alpha$ is

$$S \;=\; S_0(\phi) \;+\; \phi^*_i {R^i}_\alpha(\phi)c^\alpha \;-\; \tfrac{1}{2}c^*_\gamma {f^\gamma}_{\alpha\beta}(\phi)c^\alpha c^\beta \;+\; \dots, \tag{19}$$

where the dots denote terms required when the gauge algebra closes only on-shell. The master equation (11) for (19) unpacks, order by order in antifield number, into the statement that S_0 is gauge invariant, that the ${R^i}_\alpha$ generate a (possibly open) algebra with structure functions ${f^\gamma}_{\alpha\beta}$, and the higher Jacobi-type constraints on the structure functions [1, 3].

5 The Koszul–Tate resolution and the classical master equation

The BRST differential imposes gauge invariance. Its partner, the Koszul–Tate differential, imposes the equations of motion homologically, resolving the critical locus.

Definition 5.1 (Koszul–Tate differential)**.** Let $S_0 \in \mathcal{O}(X)$ with critical ideal $I = (\partial_i S_0) \subset \mathcal{O}(X)$ generated by the equations of motion. Introduce antifields ϕ_i^* with antifield number $\mathrm{afn}(\phi_i^*) = 1$ and pure ghost number 0. The *Koszul–Tate differential* δ acts by

$$\delta\phi_i^* \;=\; -\frac{\partial S_0}{\partial\phi^i}, \qquad \delta\phi^i = 0, \tag{20}$$

extended as a derivation of antifield number -1. For reducible theories one adds higher antifields ϕ^{**} (antighosts of antighosts) to kill the relations among the $\partial_i S_0$; their number equals the number of nontrivial Noether identities.

Theorem 5.2 (Koszul–Tate resolves the on-shell functions)**.** *If the critical locus* $\mathrm{Crit}(S_0) = \{\partial_i S_0 = 0\}$ *has the expected dimension (the* $\partial_i S_0$ *form a regular sequence, up to Noether relations which the higher antifields resolve), then*

$$H_k(\delta) \;=\; \begin{cases} \mathcal{O}(X)/I = \mathcal{O}(\mathrm{Crit}(S_0)) & k = 0, \\ 0 & k > 0. \end{cases} \tag{21}$$

That is, the Koszul–Tate complex is a free resolution of the on-shell function ring $\mathcal{O}(\mathrm{Crit}(S_0))$ *as an* $\mathcal{O}(X)$*-module.*

Proof. For a regular sequence $(g_1, \dots, g_r) = (\partial_1 S_0, \dots)$, the Koszul complex $\Lambda^\bullet(\bigoplus \mathcal{O}(X)\phi_i^*)$ with differential $\delta\phi_i^* = -g_i$ has homology concentrated in degree 0, equal to $\mathcal{O}(X)/(g_i)$; this is the classical acyclicity of the Koszul complex on a regular sequence. When the g_i fail to be regular precisely because of Noether identities $\sum_i Z_a^i\,\partial_i S_0 = 0$ (which for a gauge theory are forced by gauge invariance, $Z_a^i \propto R_a^i$), one adjoins second-generation antifields C_a^* with $\delta C_a^* = Z_a^i \phi_i^*$ to cancel the spurious degree-one homology. Iterating over the tower of reducibility relations produces a resolution with $H_0 = \mathcal{O}(\mathrm{Crit}(S_0))$ and $H_{k>0} = 0$; this is the Tate construction adjoining generators to kill cycles [3, Ch. 8–10], [4]. [S] □

5.1 Assembly: the full BV complex as a bicomplex

The BRST differential s (pure ghost number $+1$, antifield number 0) and the Koszul–Tate differential δ (pure ghost number 0, antifield number -1) assemble, together with correction terms, into the single BV differential $Q_{\mathrm{BV}} = (S, -)$ of total ghost number $+1$. Expanding in antifield number,

$$Q_{\mathrm{BV}} \;=\; \delta \;+\; s \;+\; (\text{higher-order in antifields}). \tag{22}$$

The bigrading is displayed in fig. 1. The horizontal direction is pure ghost number (increased by s); the vertical is antifield number (decreased by δ). Q_{BV} moves one step up-and-right in total ghost number.

$$
\begin{tikzcd}
\vdots \arrow[d, dash] & \vdots & \vdots \\
\mathcal{O}\{\phi^*\phi^*\} \arrow[r, "s"] \arrow[d, "\delta"'] & \cdot \arrow[r, "s"] & \cdot \\
\mathcal{O}\{\phi^*\} \arrow[r, "s"] \arrow[d, "\delta"'] & \mathcal{O}\{\phi^* c\} \arrow[r, "s"] & \cdot \\
\mathcal{O}(X) \arrow[r, "s"'] & \mathcal{O}\{c\} \arrow[r, "s"'] & \mathcal{O}\{cc\}
\end{tikzcd}
$$

Figure 1: The classical BV bicomplex. Rows are graded by pure ghost number (horizontal, s); columns by antifield number (vertical, δ). The total differential $Q_{\mathrm{BV}} = \delta + s + \cdots$ raises total ghost number $\mathrm{gh} = \mathrm{pgh} - \mathrm{afn}$ by one. Physical observables sit in the total-degree-zero cohomology at the bottom-left corner: δ-closed mod exact (on-shell) and s-closed (gauge invariant).

Theorem 5.3 (Observables from the bicomplex, standard spectral sequence)**.** *Assume Koszul–Tate acyclicity (theorem 5.2) and that the gauge generators close. Then the cohomology of $Q_{\mathrm{BV}} = \delta + s + \cdots$ in total ghost number 0 is computed by the spectral sequence of the bicomplex and equals*

$$H^0(Q_{\mathrm{BV}}) \;=\; \big(\mathcal{O}(\mathrm{Crit} S_0)\big)^{\mathfrak{g}} \;=\; \mathcal{O}(\mathrm{Crit} S_0 /\!/ G), \tag{23}$$

the gauge-invariant functions on the critical (on-shell) locus.

Proof. Filter by antifield number. The first page of the spectral sequence takes δ-cohomology, which by theorem 5.2 is $\mathcal{O}(\mathrm{Crit} S_0)$ concentrated in antifield number 0. The differential induced on the first page is s, restricted to on-shell functions, whose degree-zero cohomology is the invariants by theorem 4.3. Because δ-cohomology is concentrated in a single antifield degree, the spectral sequence degenerates at the second page and $H^0(Q_{\mathrm{BV}}) = (\mathcal{O}(\mathrm{Crit} S_0))^{\mathfrak{g}}$ with no further corrections [3, Ch. 8], [4]. `[S]` □

Remark 5.4 (Open gauge algebras). The hypothesis that the gauge generators close off-shell is convenient but not essential. For an *open* algebra, where $[\delta_\alpha, \delta_\beta] = f^{\gamma}{}_{\alpha\beta}\delta_\gamma + M^{ij}_{\alpha\beta}\, \partial_j S_0\, (\partial/\partial\phi^i)$ closes only modulo the equations of motion, the higher-order-in-antifield terms of (22) are exactly the M-dependent corrections that the master equation forces. The filtration argument still applies: the E_0 page is Koszul–Tate, which is insensitive to the open terms (they carry positive antifield number and act trivially on $E_1 = \mathcal{O}(\mathrm{Crit} S_0)$), and the E_1 differential is the strict BRST differential s acting on the on-shell functions. So the spectral sequence again degenerates at E_2 and theorem 5.3 holds verbatim; the ability to handle open algebras in this way is one of the historical motivations for the BV extension over bare BRST [1, 3]. `[S]`

6 The derived critical locus and shifted symplectic structure

Theorem 5.3 is the classical statement; its clean modern home is derived algebraic geometry. We record the dictionary.

6.1 Derived critical locus

Let X be a smooth affine scheme (or the formal/local model of a space of fields) and $S \in \mathcal{O}(X)$ a function. The ordinary critical locus is $\mathrm{Crit}(S) = \{dS = 0\}$, the intersection of the graph $\Gamma_{dS} \subset T^*X$ with the zero section $X \subset T^*X$. When this intersection is not transverse, which is exactly the gauge-theoretic situation, the honest scheme $\mathrm{Crit}(S)$ loses information. The derived intersection retains it.

Definition 6.1 (Derived critical locus)**.** The *derived critical locus* of S is the homotopy fibre product

$$\mathrm{dCrit}(S) \;=\; X \times^h_{T^*X} X, \tag{24}$$

where both maps $X \to T^*X$ are the zero section and the graph dS respectively. Its structure sheaf is the Koszul complex

$$\mathcal{O}(\mathrm{dCrit} S) \;=\; \big(\mathrm{Sym}_{\mathcal{O}(X)}(\mathbb{T}_X[1]),\, \iota_{dS}\big), \tag{25}$$

the symmetric algebra on the shifted tangent complex with differential contraction against dS. In coordinates this is exactly the Koszul–Tate complex (20) with antifields ϕ_i^* generating $\mathbb{T}_X[1]$ and $\delta = \iota_{dS}$.

The square

$$\begin{tikzcd} \mathrm{dCrit}(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "0"] \\ X \arrow[r, "dS"'] & T^*X \end{tikzcd} \tag{26}$$

is a homotopy pullback. Taking π_0 (classical truncation) recovers $\mathrm{Crit}(S)$, while the higher homotopy encodes the failure of transversality, which is the gauge/ghost data.

Theorem 6.2 (PTVV; (-1)-shifted symplectic structure)**.** *The derived critical locus* $\mathrm{dCrit}(S)$ *carries a canonical* (-1)*-shifted symplectic structure. The induced* (-1)*-shifted Poisson bracket on* $\mathcal{O}(\mathrm{dCrit} S)$ *coincides, under the identification of theorem 6.1, with the BV antibracket* (7)*.*

Proof sketch. This is a special case of [5, Thm. 2.12 and §2.2]: the cotangent T^*X carries a canonical 0-shifted symplectic form, and the derived intersection of two Lagrangians (here the zero section and the graph of an exact one-form dS, both Lagrangian in T^*X) inherits a (-1)-shifted symplectic form. The graph of dS is Lagrangian precisely because dS is closed. Unwinding the construction in coordinates, the shifted symplectic pairing between the degree-0 generators ϕ^i and the degree-(-1) generators ϕ_i^* is the constant pairing whose Poisson bracket is (7); see [6] for the explicit comparison with the classical BV antibracket and for the reduction statement when S is G-invariant. [S] □

Remark 6.3 (Gauge symmetry as a stacky quotient before critical locus). When S is G-invariant one takes the derived critical locus on the quotient stack: $\mathrm{dCrit}(S)$ on $[X/G]$, equivalently the derived critical locus of the induced function on the orbit stack. Anel and Calaque [6] show that the derived symplectic reduction of $\mathrm{dCrit}(S)$ by a Hamiltonian G-action agrees with the derived critical locus of the reduced function, i.e. the two operations "impose equations of motion" (critical locus / Koszul–Tate) and "quotient by gauge" (reduction / Chevalley–Eilenberg) commute up to coherent homotopy. This commutation is the geometric content of theorem 5.3's spectral-sequence degeneration. [S]

6.2 Why the derived object is the right invariant

The derived critical locus is the honest model of "the space of solutions modulo gauge, with stabilizers retained." Its tangent complex at a solution ϕ_0 is a three-term complex

$$\mathbb{T}_{\mathrm{dCrit}(S),\phi_0}: \quad \mathfrak{g} \xrightarrow{R} T_{\phi_0}X \xrightarrow{\mathrm{Hess}(S)} T^*_{\phi_0}X \xrightarrow{R^\dagger} \mathfrak{g}^*, \tag{27}$$

in degrees $-1, 0, 1, 2$: the degree -1 part is the gauge directions (ghosts), the degree 0 the field fluctuations, the degree 1 the equations-of-motion constraints (antifields), the degree 2 the Noether identities (antighosts). The cohomology of (27) at a symmetric background is where the reconstruction story becomes concrete: a nonzero H^{-1} is exactly a stabilizer (a Killing vector, for gravity), and the framework refuses to discard it. We make this precise in section 9.

7 Main results: observables as cohomology

We collect the module's central statements. Theorem 5.3 already gave the classical observable algebra; we now state the observable dictionary, the gauge-fixing independence, and the reconstruction proposition.

Theorem 7.1 (Observable dictionary)**.** *Let $(\mathcal{F}, S, Q_{\mathrm{BV}})$ be a classical BV theory with S solving the master equation. Define the algebra of* classical observables *as $\mathrm{Obs}^{\mathrm{cl}} = H^0(Q_{\mathrm{BV}})$. Then:*

1. *$\mathrm{Obs}^{\mathrm{cl}}$ is a commutative algebra, the gauge-invariant on-shell functionals, agreeing with $\mathcal{O}(\mathrm{Crit}S_0//G)$ under Koszul–Tate acyclicity.*
2. *The (-1)-shifted symplectic structure of theorem 6.2 induces on $H^\bullet(Q_{\mathrm{BV}})$ a degree $+1$ Poisson (Gerstenhaber) bracket, the classical BV bracket of observables, which on $\mathrm{Obs}^{\mathrm{cl}}$ restricted to a Cauchy data reduces to the Peierls/Poisson bracket of the physical phase space.*
3. *Observables are graded by ghost number; the physical (measurable) sector is ghost number 0, and states of nonzero ghost number are unphysical (ghosts do not appear in S-matrix elements).*

Proof. Item (1) is theorem 5.3. Item (2): the antibracket descends to cohomology because $Q_{\mathrm{BV}} = (S, -)$ is a graded derivation of the bracket and $(S, S) = 0$, so the bracket of two closed elements is closed and the bracket with an exact element is exact; the identification with the Peierls bracket on a Cauchy surface is [10, §4]. Item (3) is the standard observation that ghost number is conserved and the vacuum has ghost number 0, so nonzero-ghost states decouple from physical amplitudes [3, Ch. 15]. `[S]` for (1), (2); `[S/H]` for the physical-sector interpretation in (3), which is standard in perturbation theory but inherits the perturbative caveat. □

Theorem 7.2 (Gauge-fixing independence)**.** *Let Ψ be a gauge-fixing fermion (an odd functional of ghost number -1), and let the gauge-fixed action be obtained by the canonical transformation $\phi^*_A \mapsto \phi^*_A + \partial\Psi/\partial\phi^A$, i.e. restriction to the Lagrangian submanifold $L_\Psi = \{\phi^*_A = \partial\Psi/\partial\phi^A\}$. Then the cohomology $H^\bullet(Q_{\mathrm{BV}})$, and hence the observable algebra, is independent of Ψ: two choices Ψ, Ψ' give canonically isomorphic cohomology.*

Proof sketch. A change of gauge-fixing fermion $\Psi \to \Psi + \delta\Psi$ is generated by the Hamiltonian flow of the antibracket with an odd generator, i.e. a Q_{BV}-canonical transformation. Such transformations act on cohomology by conjugation with an invertible operator that is the identity on Q_{BV}-cohomology, because the generator is itself Q_{BV}-exact on-shell. Concretely, the difference of the two Lagrangian restrictions is a Q_{BV}-exact deformation, so the induced maps on cohomology agree; the classic argument that the partition function $\int_{L_\Psi} e^{iS/\hbar}$ is independent of Ψ up to the choice of homologous Lagrangian is the integrated form of this statement [1, 3, 15]. `[S]` □

Theorem 7.2 is the technical statement of "the redundancy is not physical." The observable is the cohomology class; the gauge-fixed representative is a choice with no invariant meaning. This is the module's version of the library-wide slogan that a physical datum is an equivalence class, not a representative.

Proposition 7.3 (Reconstruction of the on-shell gauge quotient)**.** *The pair* $(\mathrm{dCrit}(S), Q_{\mathrm{BV}})$ *reconstructs the derived on-shell gauge quotient of the classical field theory: its degree-zero cohomology is the function algebra of the physical configuration space* $\mathrm{Crit}(S_0)/\!/G$*, and its full tangent complex* (27) *retains, in negative degree, the stabilizer (isotropy) data of each solution. In particular, for a solution with isometry algebra* $\mathfrak{k} = \mathrm{Lie}(\mathrm{Stab}_G(\phi_0))$*, one has* $H^{-1}(\mathbb{T}_{\mathrm{dCrit}(S),\phi_0}) \cong \mathfrak{k}$*.*

Proof. The degree-zero statement is theorem 5.3. For the tangent complex, (27) is the linearization at ϕ_0 of the Koszul–Tate–Chevalley–Eilenberg differential; its degree -1 cohomology is $\ker(R\colon \mathfrak{g} \to T_{\phi_0}X)$, the gauge parameters that act trivially at ϕ_0, which is precisely $\mathrm{Lie}(\mathrm{Stab}_G(\phi_0)) = \mathfrak{k}$. This is the infinitesimal form of $\mathrm{Aut}_{[X/G]}(\phi_0) \simeq \mathrm{Stab}_G(\phi_0)$ (1). `[S]` for the linear-algebra statement; `[S/H]` for the identification of $\mathfrak{k}$ with the physical Killing algebra in the gravitational application, which requires the field-theoretic model. □

8 Worked examples

We compute three examples explicitly. The first is small enough that every cohomology group is written out and checked by the companion Haskell code; it is the model whose nilpotency the code verifies.

8.1 A finite abelian gauge system

Let $X = \mathbb{R}^2$ with coordinates (q^1, q^2), and let $\mathfrak{g} = \mathbb{R}$ act by translation in q^2: $\rho(T) = \partial/\partial q^2$. This is the toy of a "pure gauge direction." Introduce one ghost c (odd, pgh $= 1$), its antifield c^* (afn $= 1$), and antifields q_1^*, q_2^* (afn $= 1$).

BRST part. The action of the invariant functions is $\mathcal{O}(X)^{\mathfrak{g}} = \mathbb{R}[q^1]$, functions of q^1 alone. The BRST differential is $s\,a = c\,\partial_{q^2} a$, and $s\,c = 0$ (abelian, no structure constants). By theorem 4.3,

$$H^0(s) = \mathbb{R}[q^1], \qquad H^k(s) = 0 \ (k \geq 1), \tag{28}$$

since $\mathbb{R}[q^1, q^2] \xrightarrow{\partial_{q^2}} \mathbb{R}[q^1, q^2]$ has kernel $\mathbb{R}[q^1]$ and is surjective (every polynomial has a q^2-antiderivative), so the ghost number 1 cohomology vanishes.

Remark 8.1 (Why the higher-cohomology formula of theorem 4.3 does not apply here). The translation group $G = \mathbb{R}$ acting by $q^2 \mapsto q^2 + t$ is the additive group $\mathbb{G}_a$, which is *unipotent*, not compact or linearly reductive. The coefficient module $\mathbb{R}[q^1, q^2]$ is therefore not completely reducible: there is no equivariant projection onto the invariants $\mathbb{R}[q^1]$. Consequently the tensor-product formula $H^k(s) \cong H^k(\mathfrak{g}) \otimes \mathcal{O}(X/G)$ of theorem 4.3, which was stated only for compact/reductive G, does not apply, and indeed it would predict a spurious nonzero H^1 (since $H^1(\mathfrak{g}) = \mathfrak{g}^* \neq 0$ for the one-dimensional abelian $\mathfrak{g}$). The correct answer is $H^1(s) = 0$: $\mathbb{R}[q^1, q^2]$ is a free, hence acyclic, module for the translation action, so all higher cohomology vanishes. This is exactly the mismatch flagged after the proof of theorem 4.3, and it is a useful reminder that the reductivity hypothesis there is load-bearing. `[S]`

Dynamics. Take $S_0(q) = \frac{1}{2}(q^1)^2$, gauge invariant (it does not depend on q^2). Equations of motion: $\partial_{q^1} S_0 = q^1 = 0$, $\partial_{q^2} S_0 = 0$. The nontrivial Koszul–Tate action is $\delta q_1^* = -q^1$, $\delta q_2^* = 0$. The critical locus is $\{q^1 = 0\} \cong \mathbb{R}$ (the q^2-axis), and

$$\mathcal{O}(\mathrm{Crit} S_0) = \mathbb{R}[q^1, q^2]/(q^1) = \mathbb{R}[q^2]. \tag{29}$$

Since $\delta q_2^* = 0$ identically, q_2^* is a δ-cocycle that is not a boundary: the equation of motion $\partial_{q^2} S_0 \equiv 0$ is a Noether identity (it is the statement that S_0 is q^2-independent). To restore acyclicity we add a second-generation antifield η with $\delta\eta = q_2^*$; this is the ghost-for-ghost of the trivial reducibility $1 \cdot \partial_{q^2} S_0 = 0$. With η adjoined, $H_{k>0}(\delta) = 0$ and $H_0(\delta) = \mathbb{R}[q^2]$.

Full BV cohomology. Combining, the degree-zero BV cohomology is gauge-invariant on-shell functions:

$$H^0(Q_{\mathrm{BV}}) = \big(\mathbb{R}[q^2]\big)^{\mathfrak{g}} = \big(\mathbb{R}[q^2]\big)^{\partial_{q^2}} = \mathbb{R}, \tag{30}$$

the constants: on the critical locus $\{q^1 = 0\}$ the only surviving coordinate is q^2, and it is pure gauge, so the physical observable algebra is one dimensional. This is the expected answer: a single free "radius" coordinate q^1 that is fixed to 0 on shell, and a single gauge coordinate q^2 that carries no invariant. The example exhibits, in miniature, both a Koszul–Tate resolution with a Noether identity and a nontrivial gauge quotient.

Remark 8.2. The Noether identity $\partial_{q^2} S_0 \equiv 0$ and the ghost-for-ghost η are the finite-dimensional avatar of the reducibility that appears for gravity on a symmetric background (section 9): a symmetry direction of the action forces a relation among the equations of motion, which forces a higher antighost. [S]

8.2 A nonabelian Lie-algebra model

Let $\mathfrak{g} = \mathfrak{su}(2)$ with structure constants $f^{\gamma}{}_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma}$, acting on $X = \mathbb{R}^3$ by the vector representation, $\rho(T_\alpha) = \varepsilon_{\alpha\beta\gamma} q^\beta \partial_{q^\gamma}$ (rotations). The invariant ring is

$$\mathcal{O}(X)^{\mathfrak{g}} = \mathbb{R}[r^2], \qquad r^2 = (q^1)^2 + (q^2)^2 + (q^3)^2, \tag{31}$$

the radial functions. By theorem 4.3, $H^0(s) = \mathbb{R}[r^2]$. The higher cohomology is $H^\bullet(\mathfrak{su}(2)) \otimes \mathbb{R}[r^2]$ away from the origin; $H^\bullet(\mathfrak{su}(2)) = \Lambda(x_3)$ is an exterior algebra on a single degree-3 generator (the Cartan 3-form), reflecting $H^\bullet(\mathrm{SU}(2)) = H^\bullet(S^3)$. Thus

$$H^0(s) = \mathbb{R}[r^2], \quad H^3(s) \supset \mathbb{R}[r^2] \cdot x_3, \quad H^1(s) = H^2(s) = 0. \tag{32}$$

The companion code builds this s for $\mathfrak{su}(2)$ symbolically and checks $s^2 = 0$ on a basis of $\mathcal{O}(X) \otimes \Lambda^{\leq 3}\mathfrak{g}^*$, giving a machine confirmation of theorem 4.2 for this case.

Adding a potential. With $S_0 = \frac{1}{4}(r^2-1)^2$ (invariant), one has $\partial_{q^i} S_0 = (r^2-1)q^i$, so the critical locus is the unit sphere $\{r^2 = 1\}$ together with the origin $\{q = 0\}$. On this locus the invariant coordinate r^2 satisfies $r^2(r^2-1) = 0$, hence

$$H^0(Q_{\mathrm{BV}}) \;=\; \mathbb{R}[r^2]/\big(r^2(r^2-1)\big) \;\cong\; \mathbb{R} \oplus \mathbb{R}, \tag{33}$$

the locally constant functions on the two invariant critical values $r^2 \in \{0, 1\}$. The physical observable algebra is two-dimensional, exactly as the reconstruction picture predicts.

8.3 Comparison with the derived critical locus

For the nonabelian model, $\mathrm{dCrit}(S_0)$ on the quotient stack $[\mathbb{R}^3/\mathrm{SU}(2)]$ has $\pi_0 = \{r^2 = 1\}/\mathrm{SU}(2) \sqcup \{0\}$, a point plus the origin, and its tangent complex at a generic point of the sphere is $\mathfrak{su}(2) \to \mathbb{R}^3 \to \mathbb{R}^3 \to \mathfrak{su}(2)^*$ with $H^{-1} = 0$ (the action is locally free away from the origin) but $H^{-1} \neq 0$ at the origin (full stabilizer $\mathfrak{su}(2)$). The jump in H^{-1} at the origin is the isotropy jump that a coarse quotient would erase; the derived object records it. This is the finite model of the gravitational statement of section 9.

9 The reducible gravitational case

We now state the module's substantive physical result. Consider perturbative gravity around a fixed background metric $\bar{g}$ on a globally hyperbolic M. The gauge symmetry is infinitesimal diffeomorphism, $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$, generated by vector fields $\xi \in \mathfrak{X}(M)$, which play the role of the gauge Lie algebra $\mathfrak{g}$. The ghost is a fermionic vector field c^μ; the BV action is the Einstein–Hilbert action extended by the antifield couplings $g^*_{\mu\nu} \mathcal{L}_c g^{\mu\nu}$ and the ghost self-coupling $c^*_\mu c^\nu \partial_\nu c^\mu$ [14].

9.1 Reducibility from Killing vectors

Definition 9.1 (Killing reducibility)**.** A vector field ξ is a *Killing vector* of $(M, \bar{g})$ if $\mathcal{L}_\xi \bar{g} = 0$, that is $\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$ (we write the symmetrized covariant derivative out in full to fix the convention). The space of Killing vectors is the Lie algebra $\mathfrak{k} = \mathrm{Lie}(\mathrm{Isom}(M, \bar{g}))$ of the isometry group.

At a background with $\mathfrak{k} \neq 0$, the gauge generator $R\colon \xi \mapsto \mathcal{L}_\xi \bar{g}$ has a kernel: Killing vectors generate no field variation. This is a *reducibility* of the gauge symmetry, and it is exactly the situation theorem 5.1 and theorem 5.2 handle by introducing ghosts-for-ghosts.

Theorem 9.2 (Ghost-for-ghost tower dual to the isometry algebra)**.** *Let $(M, \bar{g})$ be a background with isometry algebra $\mathfrak{k}$. In the BV-BRST complex of perturbative gravity around $\bar{g}$, the first-generation reducibility space is canonically*

$$\ker\big(R\colon \mathfrak{X}(M) \to \Gamma(S^2 T^* M)\big) = \mathfrak{k}, \tag{34}$$

so the complex contains ghosts-for-ghosts $c_a^{(2)}$, $a = 1, \ldots, \dim \mathfrak{k}$, of pure ghost number 2, one for each independent Killing vector. Equivalently, the tangent complex (27) of $\mathrm{dCrit}(S)$ *at $\bar{g}$ has*

$$H^{-1}\big(\mathbb{T}_{\mathrm{dCrit}(S), \bar{g}}\big) \cong \mathfrak{k}. \tag{35}$$

Proof. The map $R = \mathcal{L}_{(-)} \bar{g}$ sends $\xi \mapsto \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$. Its kernel is by definition the Killing algebra $\mathfrak{k}$ (theorem 9.1). When a gauge generator has a kernel, the Noether identities of the theory are not independent: there are relations $\sum \zeta^a R = 0$ indexed by a basis of $\mathfrak{k}$, which by the Tate construction (theorem 5.2) require second-generation antifields, dual to which are the ghosts-for-ghosts of pure ghost number 2. The identification $H^{-1}(\mathbb{T}) = \ker R = \mathfrak{k}$ is theorem 7.3 specialized to gravity. This is the perturbative-quantum echo of the slice-theorem statement that the gravitational moduli stack has isotropy group $\mathrm{Isom}(M, \bar{g})$ at $\bar{g}$; see [14] for the explicit BV-BFV data on which this reducibility structure sits. `[S/H]` □

Corollary 9.3 (Schwarzschild and FLRW)**.** *For the Schwarzschild background,* $\mathrm{Isom} = \mathbb{R}_t \times \mathrm{SO}(3)$ *so* $\dim \mathfrak{k} = 4$ *and there are four ghosts-for-ghosts (one time translation, three rotations). For a spatially homogeneous and isotropic FLRW background the isometry algebra of a constant-time slice contributes six Killing vectors (three translations, three rotations of the maximally symmetric slice), so* $\dim \mathfrak{k} = 6$ *for the spatial isometries. For Kerr,* $\mathrm{Isom} = \mathbb{R}_t \times \mathrm{U}(1)$ *and* $\dim \mathfrak{k} = 2$.

Proof. Direct from the isometry groups of these solutions and theorem 9.2. `[S]` for the isometry counts; `[H]` for the assertion that the perturbative gravitational BV complex around Kerr has been assembled explicitly, which, to our knowledge, has not been carried out in one place in the literature (see section 12). □

9.2 Interpretation

Theorem 9.2 is the sense in which "isotropy is retained, not erased" propagates from the classical moduli stack into the perturbative observable algebra. A coarse quotient of metrics by diffeomorphisms would represent Schwarzschild by a single point and forget its four-dimensional isometry group. The BV-BRST complex refuses: the four Killing vectors appear as four ghosts-for-ghosts, and they control the degeneracy structure of the linearized theory (zero modes of the kinetic operator, would-be gauge modes that are not gauge). This is the concrete mechanism behind the library's recurring pattern that physical content is a stack with retained automorphisms, not a bare quotient set.

10 Quantization: BV Laplacian, quantum master equation, obstruction

The classical results above are [S]. Quantization is where the honest status downgrades to [S/H], and we are careful to say why. Throughout this section we extend the scalars of theorem 3.1 from $\Bbbk = \mathbb{R}$ to $\Bbbk = \mathbb{C}$ (equivalently $\mathbb{R}[i]$): the path-integral weight $e^{iS_\hbar/\hbar}$ and the $i\hbar\Delta$ term in the quantum master equation below require the imaginary unit, so the graded-commutative algebra $\mathcal{O}(\mathcal{F})$ is complexified before quantization.

Definition 10.1 (BV Laplacian and quantum master equation)**.** On the doubled space, define the second-order operator

$$\Delta \;=\; (-1)^{\varepsilon_A}\,\frac{\partial^R}{\partial\phi^A}\frac{\partial^R}{\partial\phi^*_A}, \tag{36}$$

the *BV Laplacian*, of ghost number $+1$ and $\Delta^2 = 0$. A quantum action $S_\hbar = S + \hbar S_1 + \hbar^2 S_2 + \cdots$ satisfies the *quantum master equation* if

$$\tfrac{1}{2}(S_\hbar, S_\hbar) - i\hbar\,\Delta S_\hbar \;=\; 0, \qquad \text{equivalently} \qquad \Delta\, e^{iS_\hbar/\hbar} = 0. \tag{37}$$

Proposition 10.2 (Quantum BRST differential)**.** *If $S_\hbar$ solves the quantum master equation* (37)*, the operator*

$$Q_\hbar = (S_\hbar, -) - i\hbar\,\Delta \tag{38}$$

is nilpotent, $Q_\hbar^2 = 0$, and its degree-zero cohomology is the algebra of quantum observables.

Proof. Expand $Q_\hbar^2 F = (S_\hbar,(S_\hbar,F)) - i\hbar\,\Delta(S_\hbar,F) - i\hbar(S_\hbar,\Delta F) - \hbar^2\Delta^2 F$. The first term is $\frac{1}{2}((S_\hbar,S_\hbar),F)$ by graded Jacobi; the middle two combine using the fact that Δ is a *first-order* graded derivation of the antibracket,

$$\Delta(F,G) = (\Delta F, G) + (-1)^{\varepsilon_F+1}(F,\Delta G), \tag{39}$$

so that $-i\hbar\,\Delta(S_\hbar,F) - i\hbar(S_\hbar,\Delta F) = -i\hbar\,(\Delta S_\hbar, F)$; the second-order character of Δ shows up only in its failure to be a derivation of the commutative product (the extra term in $\Delta(FG)$), not of the bracket, so no (F,G)-type term appears in (39). The last term vanishes as $\Delta^2 = 0$. Collecting, $Q_\hbar^2 = (\frac{1}{2}(S_\hbar,S_\hbar) - i\hbar\Delta S_\hbar, -)$, which vanishes by (37). [S] ☐

Theorem 10.3 (Costello–Gwilliam quantization obstruction, restated)**.** *A classical BV theory admits a quantization, i.e. a solution $S_\hbar$ of the quantum master equation deforming S, if and only if a sequence of obstruction classes $O_n \in H^1(Q_{\mathrm{BV}})$, $n \geq 1$, vanishes order by order in $\hbar$. When they vanish, the set of quantizations is a torsor over $H^0(Q_{\mathrm{BV}})[\![\hbar]\!]$; the obstruction O_1 is the one-loop anomaly.*

Proof sketch. This is the factorization-algebra quantization theorem of Costello and Gwilliam [9, Vol. 2], built on Costello's renormalization [7]. Obstruction theory for solving (37) order by order in $\hbar$: given a solution mod $\hbar^n$, the failure at order $\hbar^n$ is a Q_{BV}-cocycle of ghost number $+1$ whose class $O_n \in H^1(Q_{\mathrm{BV}})$ is the obstruction to extending to order $\hbar^{n+1}$; if $[O_n] = 0$ one can correct S_n, and the ambiguity in the correction is a ghost-number-zero cocycle, i.e. an element of $H^0(Q_{\mathrm{BV}})$. [S] for the mathematics; [S/H] for the identification of $[O_1] \neq 0$ with a physical gauge anomaly, which is standard but rests on the perturbative renormalization framework. □

This is the precise, citable sense in which "renormalizable and anomaly-free" becomes "a cohomology class vanishes." It is also the precise place the module's status downgrades: the classical observable algebra $H^0(Q_{\mathrm{BV}})$ is [S], but its survival as the observable algebra of the interacting quantum theory is contingent on $H^1(Q_{\mathrm{BV}})$-valued obstructions vanishing, which is a perturbative, renormalization-scheme-dependent statement, hence [S/H].

11 Composition with the other eleven regimes

The library is modular. Its regimes compose by the weakest-link status calculus of theorem 2.2, and this section makes the compositions explicit for the gauge layer. The dictionary status of the BV-BRST/derived-critical-locus entry is [S/H] (assumptions: perturbative or derived local model; limitations: nonperturbative global completion is model dependent).

11.1 Within QG-II: factorization algebras

The natural continuation is the factorization-algebra regime, which globalizes the single-background BV complex into a local-to-global assignment of observable complexes $U \mapsto \mathrm{Obs}(U)$ over spacetime, with $H^0(\mathrm{Obs}(U))$ recovering the gauge-invariant observables on the region U [8]. Both entries are [S/H]; their composite for perturbative, local gauge theory remains [S/H]. Composed with any claim of a fully background-independent factorization algebra of quantum gravity, the composite degrades to [H], because no such construction is established.

11.2 Down to QG-I: the classical moduli stack

The classical regime represents $\mathrm{Sol(Ein)}//\mathrm{Diff}$ as a stack with isotropy $\mathrm{Isom}(M, \bar{g})$. Theorem 9.2 is the BV-level image of this. Both are [S/H]; the bridge claim "the derived critical locus is the perturbative model of the classical moduli stack" is itself [S/H] locally (theorem 6.3) but only [H] as a global 4D statement, since a global derived-stack construction of $\mathrm{Sol(Ein)}//\mathrm{Diff}$ in four dimensions is not settled (section 12).

11.3 Across to QG-III and QG-VII: the perturbative wall

Composing the gauge layer with the spin-foam regime (status [H], genuinely nonperturbative) or the asymptotic-safety regime ([H]) yields, by theorem 2.2,

$$\mathrm{compose}(\mathsf{S/H}, \mathsf{H}) = \mathsf{H}. \tag{40}$$

The weak link is BV-BRST's own limitation clause, "nonperturbative global completion is model dependent." This is not pessimism about the mathematics; it is the statement that a chain reaching a nonperturbative regime cannot advertise better than heuristic status even if the BV part is rigorous.

11.4 The sharpest case: QG-X, indefinite causal order

The proposed causal-order regime carries status [H/P]: the process-matrix mathematics is well-defined ([H]) but the claim that quantum gravity dynamically realizes indefinite causal order is [P]. Composing,

$$\mathrm{compose}(\mathsf{S}/\mathsf{H}, \mathsf{H}/\mathsf{P}) = \mathsf{P}. \tag{41}$$

A paper attempting "BV quantization of gravity on an indefinite causal background" must present its conclusion as speculative regardless of how careful the BV construction is, because the worst warrant in the chain is P. This is the monotonicity of theorem 2.2 doing exactly the work it is meant to do: bounding claims from above. The composition is verified in the companion code by exhaustive evaluation over the status set.

11.5 Summary table

Composition	Bridge claim	Composite status
QG-II ∘ QG-II	local factorization algebra of gauge theory	S/H
QG-II ∘ QG-I	derived crit. locus = perturbative moduli stack	S/H (H global)
QG-II ∘ QG-III	BV gravity meets spin-foam dynamics	H
QG-II ∘ QG-VII	BV gravity meets asymptotic safety	H
QG-II ∘ QG-X	BV gravity on indefinite causal order	P

12 Limitations and open problems

We are explicit about what this module does not establish.

1. **Perturbative only.** The entire construction is perturbative or, at best, formal/local in the derived-geometric sense. The Costello–Gwilliam quantization theorem lives in perturbative renormalization; there is no nonperturbative, background-independent BV complex for quantum gravity. Every claim that reaches beyond the perturbative regime is tagged [H] or worse.

2. **Kerr ghost-for-ghost not carried out.** Theorem 9.3 counts ghosts-for-ghosts by isometry dimension, which is rigorous, but a fully explicit assembly of the perturbative gravitational BV complex around Kerr (rotating, lower symmetry, where the reducibility interacts with the ergoregion and horizon structure) is, to our knowledge, not written down in one place. The closest available starting points are the BV-BFV data for the Einstein–Hilbert action of Cattaneo and Schiavina [14] and the boundary/corner axioms of Cattaneo, Mnev and Reshetikhin [13], neither of which specializes the reducibility structure to a stationary axisymmetric background with a horizon. We flag the Kerr assembly as the concrete original computation this module invites. [H]

3. **Global 4D derived moduli stack.** A global (not merely local/formal) construction of Sol(Ein)//Diff as a derived stack in four dimensions, accounting for the non-compact diffeomorphism group, isotropy jumps, and matter coupling, is not settled. The 2019 "Stacks in Einstein Gravity" construction is 3D and Chern–Simons-equivalent; a 4D follow-up matching the generality this library wants is not established. `[H/P]`

4. **Cut/regulator complementarity.** Whether the BV-BRST cohomology differential and the coaction/cobracket decomposition of the amplitude regime are literally adjoint functors in a suitable category is open (roadmap item 2); we state it as a structural analogy, not a theorem. `[P]`

5. **Formalization gap.** The companion Haskell code checks nilpotency for finite models by exhaustive evaluation. A machine-checked proof of $Q_{\mathrm{BV}}^2 = 0$ for the general BV complex (not a fixed finite model), in a proof assistant, remains future work (roadmap item 7). The finite check is `[S]`; the general formal proof is not yet done.

13 Conclusion

The gauge layer of the reconstruction program has a clean invariant: observables are the degree-zero cohomology of the BV differential $Q_{\mathrm{BV}} = (S, -)$ on the derived critical locus $\mathrm{dCrit}(S)$. The derived critical locus is the homotopical model of the on-shell gauge quotient; it carries the PTVV (-1)-shifted symplectic structure whose bracket is the BV antibracket; its degree-zero cohomology is the gauge-invariant on-shell functionals; and its tangent complex retains, in negative degree, exactly the stabilizer data (Killing vectors, for gravity) that a coarse quotient would erase. The reconstruction reading is not decorative: gauge redundancy is not noise to be removed before quantization but invariant representation structure whose cohomology reconstructs what can be measured.

We have been careful to tag each step. The classical cohomological statements are standard mathematics (`[S]`). Their survival into the interacting quantum theory is contingent on a vanishing obstruction in $H^1(Q_{\mathrm{BV}})$ and is therefore heuristic and perturbative (`[S/H]`). Any extension to a nonperturbative, background-independent, or indefinite-causal-order regime is speculative and is bounded, not asserted, by the weakest-link status calculus. The two computations we offer as our own contribution, the fully explicit finite BV model with its machine-checked nilpotency, and the ghost-for-ghost tower dual to the isometry algebra on a symmetric gravitational background, are both firmly in the `[S]`/`[S/H]` range, and they are exactly the pieces that make the slogan "physical content is a stack, not a quotient" concrete at the level of perturbative quantum gravity. What remains open, the Kerr computation, the global 4D derived moduli stack, the adjunction with the amplitude regime, and the proof-assistant formalization, is stated plainly as open.

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This work is part of the YonedaAI modular quantum-gravity representation library. It reuses the shared status calculus and dictionary of that project and cites external literature verified

against arXiv and journal metadata.

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