A Fixed Point in Theory Space: Asymptotic Safety as the Renormalization-Group Completion Regime of a Modular Reconstruction Program

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Abstract

We treat asymptotic safety as one regime of a modular program in which spacetime, matter, causality, and measurement are reconstructed from invariant representation structures rather than obtained by quantizing fields on a fixed background. In this regime the invariant is not a geometric or algebraic object attached to a single action. It is a feature of the renormalization-group flow on the space of all diffeomorphism-invariant actions: an interacting (non-Gaussian) ultraviolet fixed point together with its universality class, the finite-dimensional surface of trajectories it captures as the cutoff is removed. We assemble the machinery that makes this precise. We state the Wetterich functional renormalization-group equation for the effective average action, define theory space, the beta-function vector field, fixed points, the linearized stability matrix, critical exponents, and the ultraviolet-critical surface, and we isolate the one clean rigorous fact of the subject: at any non-Gaussian fixed point of the Einstein–Hilbert flow the *background* graviton anomalous dimension is fixed to $\eta_N^* = 2 - d$, independent of scheme (in bimetric setups the fluctuation graviton's anomalous dimension is a distinct object). We work the Einstein–Hilbert truncation in $d = 4$ with the optimized cutoff in full, writing the beta functions as explicit rational functions of the dimensionless Newton coupling g and cosmological constant λ, locating the Reuter fixed point at $(g_*, \lambda_*) \approx (0.707, 0.193)$ with universal product $g_*\lambda_* \approx 0.137$, and diagonalizing the 2×2 stability matrix to obtain the complex pair of critical exponents $\theta = 1.475 \pm 3.043\, i$, so that the Einstein–Hilbert ultraviolet-critical surface is two-dimensional. Every number here is reproduced by a small base-only Haskell program that finds the fixed point, verifies $\eta_N^* = -2$ to machine precision, and computes the eigenvalues. We place the

regime inside a weakest-link status calculus and compose it honestly with the spin-foam continuum-limit regime and the classical Lorentzian-geometry regime, noting where no bridge exists. We are candid about what is not settled: individual fixed-point coordinates are scheme dependent; the evidence for a finite (≈ 3) number of relevant directions rests on truncation "apparent convergence" rather than a theorem; Euclidean-to-Lorentzian continuation, unitarity in the presence of higher-derivative operators, and full background independence are open. The composite status of the regime is H, set by the fixed-point dictionary entry, not by the functional-RG mathematics, which is S.

Contents

1 Introduction

1.1 The reconstruction thesis, applied to a fixed point

The program this paper belongs to takes a single governing statement seriously across twelve regimes of quantum gravity:

> Quantum gravity is not primarily the quantization of material objects in spacetime. It is the reconstruction of spacetime, matter, causality, and measurement from invariant representation structures.

In most regimes the invariant is something one can point to inside a single description: a boundary Hilbert space carrying representations of $SU(2)$, a moduli stack of solutions with its isotropy groups, a code subspace inside a boundary algebra. Asymptotic safety is the regime where this reading is least obvious and, we argue, most instructive, because the invariant lives one level up. There is no single action that carries it. The object that decides whether a quantum theory of the metric exists at all energies, and how many low-energy numbers one must measure to fix it, is a feature of the flow on the space of all actions: an interacting ultraviolet fixed point and the geometry of the trajectories it controls.

Weinberg named the scenario in 1976 and set it out in 1979 [1]. A field theory is *asymptotically safe* if its renormalization-group (RG) trajectories, traced toward the ultraviolet, approach a fixed point of the dimensionless couplings with a finite number of ultraviolet-attractive directions. Asymptotic freedom is the special case where that fixed point is the free (Gaussian) one. The proposal for gravity is that the fixed point is instead *interacting*: a non-Gaussian fixed point (NGFP), now called the Reuter fixed point after the paper that first computed the gravitational flow with functional methods [3]. If such a fixed point exists with a finite-dimensional attractive surface, gravity is nonperturbatively renormalizable and predictive, with only finitely many free parameters, even though it is perturbatively nonrenormalizable around the Gaussian point.

1.2 Why the invariant is truncation-invariant data, not a coupling

The reason this regime sharpens the reconstruction thesis is that the natural coordinates on theory space, the truncations one is forced to use, are exactly the non-invariant data. One never solves the exact flow. One projects it onto a finite-dimensional subspace of actions (the Einstein–Hilbert subspace g, λ; polynomial $f(R)$ subspaces; higher-derivative subspaces) and asks whether the fixed point and its critical exponents stabilize as the subspace grows. The individual fixed-point coordinates g_*, λ_* shift when one changes the regulator shape or the gauge. The would-be physical outputs, the combinations that should not depend on those choices, are the universal ones: the product $g_*\lambda_*$, the critical exponents θ_i, and above all the dimension of the ultraviolet-critical surface. This is the same discipline the modular program enforces everywhere through its stack-theoretic packaging, translated into RG language: physical content is the invariant of the flow, and "coordinate dependence" here means "truncation and scheme dependence."

We therefore keep two ledgers throughout. One records the standard, checkable mathematics of the functional RG (status S): the Wetterich equation, the beta functions of a given truncation, the fixed-point equation, the diagonalization of the stability matrix. The other records the physical hypothesis (status H): that the fixed point found in truncations is the projection of a genuine fixed point of the exact flow, that it survives Lorentzian continuation, and that the resulting theory is unitary. The dictionary entry for this regime carries the bare label H, and we do not upgrade it.

1.3 What this paper does

Section 2 states the shared substrate: the representation-stack pipeline, the $\mathsf{S}/\mathsf{H}/\mathsf{P}$ status calculus, and the dictionary table, so the paper is self-contained. Section 3 builds the framework: theory space, the effective average action, the Wetterich equation, dimensionless couplings, beta functions, fixed points, the stability matrix, critical exponents, the ultraviolet-critical surface, and the notion of predictivity. Section 4 states the paper's results as labeled propositions with proofs or proof sketches, each tagged with its epistemic status. The clean rigorous statement is Theorem 4.2: the background graviton anomalous dimension at any Einstein–Hilbert NGFP equals $2 - d$. Section 5 works the Einstein–Hilbert truncation completely, with closed-form beta functions, the Reuter fixed point, the 2×2 stability matrix, and the critical exponents, cross-checked against the literature and against our own code. That worked example is a pedagogical reproduction of Litim's 2004 optimized-cutoff computation [8], used here as a concrete testbed for the epistemic calculus rather than as a new functional-RG result: the contribution of this paper is the structural and epistemic framing, not new beta functions. Section 6 places the regime in the weakest-link calculus and composes it with the spin-foam and classical-geometry regimes, flagging where bridges are missing. Section 7 is a candid account of the open problems: truncation and scheme dependence, Euclidean versus Lorentzian, unitarity and higher-derivative ghosts, background independence. Section 8 documents the accompanying Haskell program. Sections 9 and 10 discuss and conclude.

1.4 What we claim and what we do not

We claim the mathematics of the truncated flow, as stated, and the specific numbers of the Einstein–Hilbert example, which we reproduce independently in code. We claim the epistemic bookkeeping: which statements are theorems, which are computations in a truncation, and which are physical hypotheses. We do not claim that asymptotic safety is established. The honest summary, which we defend in Section 7, is that the existence of the Reuter fixed point is very well supported inside Euclidean truncations and by the "apparent convergence" of the $f(R)$ bootstrap, while its truncation independence, Lorentzian realization, and unitarity remain open. The composite status of the regime is H.

2 The modular reconstruction program and the status calculus

This paper is one installment of a twelve-paper program, each treating a different regime of quantum gravity as a chapter in a shared modular framework. This section reproduces the shared substrate so the present paper stands alone; readers of the sibling papers may skim to Section 3. The dictionary table below is the common reference of all twelve papers, and only one of its twenty-six rows, the highlighted "RG fixed point in theory space" entry, is the direct subject here; we keep the full table so the reader can see where this regime sits among the others.

2.1 The representation-stack pipeline

A representation entry is a tuple $E = (M, P, \tau, \sigma)$ of a mathematical object M, a physical target P, a translation datum τ, and an epistemic status $\sigma \in \{\mathsf{S}, \mathsf{H}, \mathsf{P}\}$. Observables are produced by a realization pipeline

$$\begin{gathered} M \longrightarrow \Phi(M) \longrightarrow \mathrm{Real}_\alpha(\Phi(M)) \longrightarrow \text{physical representation} \longrightarrow \mathrm{Obs}, \\ \mathrm{Obs}_\alpha(M) = \mathrm{Obs}\big(\mathrm{Real}_\alpha(\Phi(M))\big). \end{gathered} \tag{1}$$

The program's structural claim is that physical content is invariant data of this pipeline, not any single coordinate-dependent presentation, and that local data glue to global data by equivalence (retaining automorphism/isotropy information) rather than by strict equality. In the RG regime the pipeline reads: theory space $\mathcal{T}$ (the space of actions) plays the role of M; the RG flow plays the role of Φ; a truncation plays the role of a realization Real_α;

and the observable is universal fixed-point data (critical exponents, the dimension of the ultraviolet-critical surface). The invariance we care about is invariance under change of truncation and scheme.

2.2 The S/H/P calculus

Three ordered warrants encode how much a translation is trusted: $\mathsf{S} < \mathsf{H} < \mathsf{P}$, where S is standard mathematical/mathematical-physics correspondence, H is a strong heuristic physical representation, and P is a speculative ontological extension. A status is a nonempty set of warrants, displayed by its range (S, S/H, H, and so on). Composition is monotone and worst-component-wins.

Definition 2.1 (Status composition)**.** For statuses a, b with worst warrants $w(a), w(b)$, define $a \circ b := \{\max(w(a), w(b))\}$. Composition is associative and commutative on worst components, is monotone (refining either argument cannot improve the composite), and has S as unit. Consequently a claim assembled by chaining regimes carries the status of its weakest link.

This is the formal engine of Section 6. It bounds a composite from above: no chain built from an H regime and an S/H regime can be better than H, regardless of how suggestive the physical story is.

2.3 The dictionary table

Table 1 reproduces the program's 26-entry translation table. The row governing this paper is highlighted.

The entry *RG fixed point in theory space* $\to$ *asymptotically safe UV completion* is one of three bare-H entries (with spin foams and celestial data). Its assumptions clause reads "truncation or continuum limit supports the fixed point" and its limitation clause reads "evidence and universality depend on truncations, observables, and Lorentzian continuation." We honor both throughout.

3 Mathematical framework

3.1 Theory space

Definition 3.1 (Theory space)**.** Fix a field content (here the metric $g_{\mu\nu}$, together with the gauge-fixing and ghost sectors, and optionally matter). *Theory space* $\mathcal{T}$ is the space of action functionals invariant under the field content's symmetry (diffeomorphisms), each written as a possibly infinite sum

$$\Gamma[\,\cdot\,] \;=\; \sum_i \bar{u}_i\, \mathcal{O}_i[\,\cdot\,], \tag{2}$$

over a basis $\{\mathcal{O}_i\}$ of diffeomorphism-invariant operators (e.g. $\int\sqrt{g}$, $\int\sqrt{g}\,R$, $\int\sqrt{g}\,R^2$, $\int\sqrt{g}\,R_{\mu\nu}R^{\mu\nu}$, and so on), with dimensionful couplings $\bar{u}_i$. A point of $\mathcal{T}$ is a choice of all the $\bar{u}_i$.

Table 1: The modular quantum-gravity dictionary (26 entries). The highlighted row, “RG fixed point in theory space,” is the subject of this paper.

St.	Math	Physical representation	Tags
[S]	Representation entry	status-labeled dictionary arrow	core
[S/H]	Representation prestack / stack	local-to-global representation library	descent
[S/H]	Realization pipeline	observable-producing computation	pipeline
[S/H]	Motive / period datum	pre-numerical object behind an observable	periods
[S/H]	Lie coalgebra / Hopf coaction	decomposition law of observables	coalgebra
[S]	Gauge groupoid / quotient stack	many representatives, one content	isotropy
[S/H]	Lorentzian manifold with metric	classical spacetime configuration	classical-limit
[S]	Diffeomorphism groupoid of fields	gauge-equivalence class of geometries	gauge
[S/H]	Moduli stack of Einstein solutions	classical possibilities with isotropy	moduli
[S/H]	Principal bundle with connection	gravitational gauge variables	connection
[S/H]	Spin network (graph, $SU(2)$ labels)	quantum-geometry boundary state	spin-network
[H]	Spin foam 2-complex	covariant transition amplitude	spin-foam
[S/H]	Cobordism category	spacetime process as morphism	tqft
[S/H]	BV–BRST / derived critical locus	gauge-fixed QFT with ghosts	bv-brst
[S/H]	Tensor network / isometric code	holographic bulk-boundary map	holography
[H]	Entanglement-wedge reconstruction	bulk operator on a boundary subregion	qec
[S/H]	Positive geometry, canonical form	amplitude from boundary/residue data	amplitudes
[S/H]	Color–kinematics dual numerators	gravity as gauge double copy	double-copy
[H]	Motivic coaction (Goncharov)	decomposition anatomy of amplitudes	motives
[H]	Celestial sphere conformal data	flat-space S-matrix as CFT correlator	celestial
[H]	**RG fixed point in theory space**	**asymptotically safe UV completion**	**asymptotic-safety**
[S/H]	Area/entropy functional	black-hole / holographic entropy	entropy
[H]	Causal set / locally finite	causal-discrete spacetime	causal-set

$\mathcal{T}$ is infinite dimensional. The couplings carry canonical mass dimensions $[\bar{u}_i] = d_i$; the corresponding dimensionless couplings are $u_i = k^{-d_i}\bar{u}_i$, where k is the RG scale. The RG flow is a vector field on the dimensionless theory space.

3.2 The effective average action and the Wetterich equation

The tool is the effective average action Γ_k, a scale-dependent functional that includes the effect of fluctuations with momenta above k and freezes those below. One adds to the bare action a regulator (cutoff) term $\Delta S_k[\varphi] = \frac{1}{2}\int \varphi\, \mathcal{R}_k\, \varphi$, quadratic in the fluctuation field φ, with a kernel $\mathcal{R}_k(p^2)$ that behaves as k^2 for $p^2 \ll k^2$ (giving low modes a large mass, suppressing them) and vanishes for $p^2 \gg k^2$ (leaving high modes untouched). The Legendre transform of the regulated generating functional, with the regulator subtracted, is Γ_k. As $k \to 0$ it becomes the ordinary effective action; as $k \to \Lambda_{\mathrm{UV}}$ it approaches the bare action.

Theorem 3.2 (Wetterich equation; Wetterich 1993, Reuter 1998)**.** **[S]** *The effective average action satisfies the exact functional flow equation*

$$\partial_t \Gamma_k \;=\; \frac{1}{2}\, \mathrm{STr}\Big[\big(\Gamma_k^{(2)} + \mathcal{R}_k\big)^{-1}\, \partial_t \mathcal{R}_k\Big], \qquad t = \ln(k/k_0), \tag{3}$$

where $\Gamma_k^{(2)}$ is the second functional derivative of Γ_k with respect to the fields, STr *is a functional supertrace (summing over field species, momenta, and internal indices, with a minus sign for Grassmann-odd ghosts), and $\partial_t \mathcal{R}_k$ is the scale derivative of the regulator.*

Proof sketch. Differentiate the regulated Schwinger functional $W_k[J] = \ln\int \mathcal{D}\varphi\, \exp(-S[\varphi] - \Delta S_k[\varphi] + J\varphi)$ with respect to t at fixed source J. Only the explicit k-dependence in ΔS_k contributes, giving $\partial_t W_k = -\frac{1}{2}\langle \varphi\, \partial_t \mathcal{R}_k\, \varphi\rangle$. Passing to the Legendre transform $\Gamma_k[\phi] = \sup_J (J\phi - W_k) - \Delta S_k[\phi]$ and using that the connected two-point function is $(\Gamma_k^{(2)} + \mathcal{R}_k)^{-1}$ yields (3). The trace is ultraviolet finite because $\partial_t \mathcal{R}_k$ is peaked at $p^2 \sim k^2$ and infrared finite because $\mathcal{R}_k$ regulates low momenta. Reuter [3] adapted this to gravity by taking φ to be the metric fluctuation about a background, adding the gauge-fixing and ghost sectors, and reading the supertrace over the background-covariant Laplacian's spectrum. Full derivations are in [2, 3, 17]. □

Two features matter for the reconstruction reading. First, (3) is exact: no expansion in a small coupling has been made. Second, it is a flow on $\mathcal{T}$: it tells the whole tower of couplings how to change with scale.

3.3 Beta functions, fixed points, stability

Projecting (3) onto a basis of operators and passing to dimensionless couplings $u_i = k^{-d_i}\bar{u}_i$ gives an autonomous system.

Definition 3.3 (Beta functions and RG flow)**.** The *beta functions* are $\beta_i(u) := \partial_t u_i$, a vector field on the dimensionless theory space. An *RG trajectory* is an integral curve $u(t)$. Toward the ultraviolet means $t \to +\infty$ ($k \to \infty$); toward the infrared means $t \to -\infty$.

Definition 3.4 (Fixed point, Gaussian and non-Gaussian)**.** A *fixed point* is u_* with $\beta_i(u_*) = 0$ for all i. It is *Gaussian* (free) if $u_* = 0$, and *non-Gaussian* (interacting) if $u_* \neq 0$.

Definition 3.5 (Stability matrix, critical exponents)**.** The *stability matrix* at a fixed point is $B_{ij} := \left.\dfrac{\partial \beta_i}{\partial u_j}\right|_{u_*}$. Linearizing the flow, $\partial_t(\delta u_i) = B_{ij}\,\delta u_j$ with solution $\delta u_i(t) = \sum_I C_I\, V_i^I\, e^{\vartheta_I t}$, where ϑ_I are the eigenvalues of B and V^I its eigenvectors. The *critical exponents* are

$$\theta_I := -\,\vartheta_I = -\,\mathrm{eig}_I(B). \tag{4}$$

The sign convention is chosen so that ultraviolet behavior is legible. As $t \to +\infty$ a perturbation grows like $e^{\vartheta_I t} = e^{-\theta_I t}$. A perturbation with $\mathrm{Re}\,\theta_I > 0$ is *suppressed* toward the ultraviolet, hence drawn into the fixed point; we call it *relevant* (ultraviolet-attractive). A perturbation with $\mathrm{Re}\,\theta_I < 0$ grows away from the fixed point toward the ultraviolet; we call it *irrelevant*. (This is the Wilsonian convention; irrelevant here means ultraviolet-repulsed, equivalently infrared-attractive.)

3.4 The ultraviolet-critical surface and predictivity

Definition 3.6 (Ultraviolet-critical surface)**.** The *ultraviolet-critical surface* $\mathcal{S}_{\mathrm{UV}}$ of a fixed point u_* is the set of points in theory space whose RG trajectory is dragged into u_* as $t \to +\infty$:

$$\mathcal{S}_{\mathrm{UV}} \;=\; \{\, u_0 \in \mathcal{T} \;:\; \lim_{t\to+\infty} u(t) = u_*, \quad u(0) = u_0 \,\}. \tag{5}$$

Its tangent space at u_* is spanned by the eigenvectors V^I with $\mathrm{Re}\,\theta_I > 0$. Hence $\dim \mathcal{S}_{\mathrm{UV}} = \#\{I : \mathrm{Re}\,\theta_I > 0\}$, the number of relevant directions.

Definition 3.7 (Asymptotic safety)**.** A theory is *asymptotically safe* at u_* if u_* is a fixed point whose ultraviolet-critical surface is finite dimensional. If u_* is non-Gaussian this is the asymptotic-safety scenario for gravity; if u_* is Gaussian it reduces to asymptotic freedom.

Predictivity is a corollary of finite $\dim \mathcal{S}_{\mathrm{UV}}$. A trajectory that reaches the ultraviolet fixed point must lie on $\mathcal{S}_{\mathrm{UV}}$. To specify it one gives its position within $\mathcal{S}_{\mathrm{UV}}$, which takes exactly $\dim \mathcal{S}_{\mathrm{UV}}$ numbers. Fixing that many low-energy observables pins the trajectory, and every other coupling at every scale is then determined.

Proposition 3.8 (Predictivity from a finite critical surface)**.** **[S/H]** *If* $\dim \mathcal{S}_{\mathrm{UV}} = n < \infty$*, then the set of asymptotically safe trajectories is an* n*-parameter family. Measuring* n *independent low-energy quantities determines the trajectory uniquely, and hence predicts all remaining couplings at all scales.*

Proof. $\mathcal{S}_{\mathrm{UV}}$ is (near u_*) an n-dimensional manifold by Theorem 3.6. The map from a trajectory to n generic low-energy observables is a local diffeomorphism from $\mathcal{S}_{\mathrm{UV}}$ onto an open set of $\mathbb{R}^n$ (its differential is generically invertible because the n observables are independent), so it is locally invertible. Inverting it recovers the point of $\mathcal{S}_{\mathrm{UV}}$, and the flow then determines $u(t)$ for all t. The status is S/H rather than S because the identification of the abstract prediction with a laboratory measurement carries the usual heuristic step of relating fixed-point couplings to physical scattering data. ∎

For contrast, at the Gaussian point the critical exponents are the canonical mass dimensions, $\theta_I = d_I$ (the classical scaling dimensions of the couplings). In $d = 4$ Newton's coupling has $\theta = d - 2 = 2$ at the free point in the sense that its dimensionless version has a marginal/irrelevant classical scaling; the perturbative nonrenormalizability of Einstein gravity is the statement that the Gaussian fixed point's critical surface does not contain the physical trajectory. Asymptotic safety replaces the Gaussian point by an interacting one whose critical surface does.

3.5 The reconstruction reading, made precise

The framework lets us say exactly what is invariant. A truncation is a choice of finite-dimensional subspace $\mathcal{T}_\alpha \subset \mathcal{T}$ and a projection of (3) onto it; a scheme is a choice of regulator shape and gauge. Both are "coordinates." The fixed-point coordinates $u_*^{(\alpha)}$ depend on α. The data we treat as physical are the ones expected to be independent of α in the exact theory: the dimension $\dim \mathcal{S}_{\mathrm{UV}}$, the critical exponents θ_I, and universal coupling combinations. The reconstruction claim is that "which quantum theory of the metric exists" is encoded in this truncation-invariant data, exactly as the program elsewhere encodes physical content in gauge/diffeomorphism-invariant data rather than in a coordinate presentation. The weak point, honestly stated, is that truncation invariance is a hypothesis here, not a theorem, which is why the regime is H.

4 Main results

We now state the paper's results. Each is tagged with its status. The rigorous statements (Theorems 4.1 and 4.2, and Theorem 4.4 read as a fact about the truncated dynamical system) are S; the elevation of the truncation result to a statement about quantum gravity (Theorem 4.5) is the asymptotic-safety hypothesis, status H.

4.1 The Gaussian point and perturbative nonrenormalizability

Proposition 4.1 (Gaussian fixed point of the Einstein–Hilbert flow)**.** **[S]** *In the Einstein–Hilbert truncation (Section 5) the origin $(g, \lambda) = (0, 0)$ is a fixed point, the Gaussian fixed point. The half-line $g = 0$ is flow-invariant, but no point $(0, \lambda)$ with $\lambda \neq 0$ is a fixed point.*

Proof. The Einstein–Hilbert beta functions (Equations (19) and (20)) have the form $\beta_g = (d - 2 + \eta_N)\, g$ and $\beta_\lambda = -(2 - \eta_N)\lambda + \frac{g}{8\pi}[\cdots]$, where the graviton anomalous dimension $\eta_N = g\, B_1/(1 - g\, B_2)$ is proportional to g at leading order and the bracket is finite at $\lambda = 0$. At $(0, 0)$, $\eta_N = 0$ and both beta functions vanish, so the origin is a fixed point. On the line $g = 0$ we have $\eta_N = 0$, so $\beta_g = (d - 2) \cdot 0 = 0$ identically: the line is invariant. But there $\beta_\lambda = -2\lambda$, which vanishes only at $\lambda = 0$. Hence $(0, \lambda)$ with $\lambda \neq 0$ is not a fixed point. This is verified numerically as property P3 of the companion code. $\square$

At the Gaussian point the critical exponents are the canonical dimensions: $\theta_g = 2 - d$ for the dimensionless Newton coupling ($= -2$ in $d = 4$, irrelevant) and $\theta_\lambda = 2$ for the dimensionless cosmological constant (relevant). The single relevant direction is not enough to accommodate a predictive theory of gravity around the free point, which is the RG face of perturbative nonrenormalizability: infinitely many operators become relevant once interactions are included around $g = 0$, or equivalently the physical trajectory does not lie on the Gaussian critical surface.

4.2 The scheme-independent fixed-point identity

The one clean theorem of the subject, which requires no numerics and no choice of scheme, is that the Newton beta function forces the graviton anomalous dimension at any interacting fixed point.

Proposition 4.2 (Anomalous dimension at a non-Gaussian fixed point)**. [S]** *Let $\beta_g = (d-2+\eta_N)\,g$ be the beta function of the dimensionless Newton coupling, where η_N is the graviton anomalous dimension. At any non-Gaussian fixed point ($g_* \neq 0$),*

$$\eta_N(g_*, \lambda_*) \;=\; 2 - d. \tag{6}$$

In $d = 4$, $\eta_N^* = -2$. This holds for every regulator, gauge, and truncation in which β_g retains the form $(d-2+\eta_N)g$, hence it is scheme independent.

Proof. At a fixed point $\beta_g(g_*, \lambda_*) = 0$, i.e. $(d-2+\eta_N^*)\,g_* = 0$. Since $g_* \neq 0$ we may divide by it, giving $d - 2 + \eta_N^* = 0$, that is $\eta_N^* = 2 - d$. The form $\beta_g = (d-2+\eta_N)g$ follows solely from the definition $g = G_k k^{d-2}$: differentiating, $\partial_t g = (d-2)g + k^{d-2}\partial_t G_k = (d-2)g + \eta_N g$, where $\eta_N := k^{d-2}\partial_t G_k/g = \partial_t \ln G_k$ up to the classical part, is the anomalous dimension. No approximation enters. The identity is verified to machine precision as property P2 of the companion code, which returns $\eta_N^* = -2.000000$ at the located fixed point. □

The identity is *kinematic*: it is a consequence of the dimensional definition $g = G_k k^{d-2}$ (which forces $\beta_g = (d-2+\eta_N)g$), not of the detailed content of the Wetterich equation. The dynamics enters only through the value of $\eta_N(g, \lambda)$ as a function on theory space; what (6) does is turn the Newton fixed-point equation into the geometric statement "the fixed point lies on the level set $\eta_N = 2 - d$." Calling it the "one clean rigorous fact" is therefore a statement about its scheme-independence and definitional certainty, not a claim that the flow equation predicts it dynamically.

Remark 4.3 (Background versus fluctuation coupling). **[S/H]** The identity (6) governs the anomalous dimension of the *background* Newton coupling, the coefficient of $\int \sqrt{g}\, R$, because it follows purely from the definition $g = G_k k^{d-2}$. In bimetric truncations that carry a separate *fluctuation* Newton coupling, the one that controls the physical graviton propagator, the fluctuation anomalous dimension is a distinct object and does not trivially satisfy this kinematic identity at the fixed point. We work in the single-metric (background) approximation throughout, where the two coincide; the distinction is one of the background-independence caveats catalogued in Section 7.

Theorem 4.2 has a physical reading worth stating: at the fixed point the dimensionful Newton coupling runs as $G_k \sim g_*/k^{d-2} = g_*/k^2$, so gravity becomes "antiscreening in exactly the borderline way" that makes the dimensionless coupling scale invariant. It is also the anchor for the numerics: the fixed-point search is really a search along the curve $\eta_N(g, \lambda) = 2 - d$ intersected with $\beta_\lambda = 0$.

4.3 Existence and content of the Reuter fixed point

Theorem 4.4 (Reuter fixed point in the Einstein–Hilbert truncation)**.** **[S]** *Consider the two-dimensional dynamical system* (19)–(20) *(the Einstein–Hilbert truncation in $d = 4$ with the optimized cutoff and standard background gauge). It has, besides the Gaussian point, a non-Gaussian fixed point (the Reuter fixed point) at*

$$(g_*, \lambda_*) \approx (0.7073,\, 0.1932), \qquad g_*\lambda_* \approx 0.1367, \qquad \eta_N^* = -2, \tag{7}$$

with a complex-conjugate pair of critical exponents

$$\theta_{1,2} = \theta' \pm i\,\theta'' \approx 1.475 \pm 3.043\,i, \qquad \theta' > 0, \tag{8}$$

so that both directions are ultraviolet-relevant and the ultraviolet-critical surface of this system is two-dimensional.

The theorem is a statement about a specific pair of rational ordinary differential equations. As such it is rigorous mathematics, status S: one exhibits the fixed point, checks $\beta = 0$, and diagonalizes the 2×2 stability matrix. We do not present a formal existence proof beyond the explicit construction in Section 5; the fixed point is located, and its properties verified, in closed form there and independently in the Haskell program of Section 8. The physical content is a separate claim, which we isolate so the ledger stays clean.

Remark 4.5 (The physical hypothesis attached to Theorem 4.4). **[H]** That the fixed point of the truncated system Theorem 4.4 is the projection of a genuine fixed point of the exact gravitational renormalization-group flow, and that it governs physical quantum gravity, is a hypothesis, not a theorem. It is the substance of the asymptotic-safety conjecture, and it carries status H. The evidence for it (truncation "apparent convergence," scheme robustness of the qualitative picture) and the reasons it is not yet established (Section 7) are discussed at length below. The number $\theta = 1.475 \pm 3.043\,i$ itself is a property of the optimized-cutoff truncation, not a measured or scheme-independent quantity.

The universal product $g_*\lambda_* \approx 0.12$–$0.14$ is comparatively stable across schemes [7, 8, 17], while the individual coordinates and the precise exponents are scheme dependent (Theorem 4.6 and section 5).

4.4 Universality and scheme dependence

Proposition 4.6 (What is universal and what is not)**.** **[H]** *In the Einstein–Hilbert truncation the individual fixed-point coordinates g_* and λ_* depend on the regulator shape and gauge and are not by themselves physical. The product $g_*\lambda_*$ and the critical exponents $\theta_{1,2}$ are more stable across schemes but still vary at the 10–30% level within this truncation. The qualitative structure, existence of an interacting fixed point with a complex ultraviolet-attractive pair and two relevant directions, is stable across all standard schemes.*

Discussion. The dependence of g_*, λ_* on the cutoff shape function and gauge parameter is explicit in comparative studies [7, 8, 10]: a sharp cutoff, an exponential cutoff, and the optimized cutoff give different coordinates but the same qualitative picture. The product

$g_*\lambda_*$ is the leading combination expected to be scheme-independent because it multiplies the on-shell action's overall scale; empirically it lands near 0.12–0.14. The critical exponents are scheme dependent at the same order in the EH truncation but converge as the truncation is enlarged (Section 7). We mark the proposition H because "universal" here is an expectation supported by computations, not a theorem about the exact flow. □

4.5 Composite status of the regime

Proposition 4.7 (Regime status)**.** **[S]** *(about the labels themselves) The asymptotic-safety regime carries composite status* H*. The functional-RG mathematics (Theorem 3.2, Theorem 4.1, Theorem 4.2, and the closed-form beta functions) is* S*; the existence-in-truncation result is a computation; the physical claim of a truncation-independent, Lorentzian, unitary fixed point is* H*; and by Theorem 2.1 the composite is the worst component,* H*.*

5 Worked example: the Einstein–Hilbert truncation

We now carry out the computation behind Theorem 4.4 in full, so that every number can be checked by hand or by the accompanying code.

5.1 Ansatz and dimensionless couplings

The Einstein–Hilbert truncation keeps two invariants, the volume term and the scalar curvature term:

$$\Gamma_k[g] \;=\; \frac{1}{16\pi G_k}\int d^dx\,\sqrt{g}\,\big(-R+2\Lambda_k\big) \;+\; \Gamma_k^{\mathrm{gf}} \;+\; \Gamma_k^{\mathrm{gh}}, \tag{9}$$

with a scale-dependent Newton coupling G_k and cosmological constant Λ_k, plus background gauge-fixing and ghost terms. The two dimensionless couplings are

$$g \;=\; G_k\,k^{d-2}, \qquad \lambda \;=\; \Lambda_k\,k^{-2}. \tag{10}$$

Inserting (9) into the Wetterich equation (3), expanding $\Gamma_k^{(2)}$ around a maximally symmetric background, decomposing the metric fluctuation into transverse-traceless, vector, and scalar parts, and reading off the coefficients of $\int\sqrt{g}$ and $\int\sqrt{g}\,R$ gives a closed two-dimensional system. We quote the result in the form due to Reuter and Saueressig [3, 7, 17].

5.2 Beta functions with threshold functions

In general d and for a general cutoff the beta functions are

$$\partial_t g = (d-2+\eta_N)\,g, \tag{11}$$

$$\partial_t\lambda = -(2-\eta_N)\lambda + \tfrac{1}{2}\,g\,(4\pi)^{1-d/2}\Big[\,2d(d+1)\,\Phi^1_{d/2}(-2\lambda) - 8d\,\Phi^1_{d/2}(0) - d(d+1)\,\eta_N\,\widetilde{\Phi}^1_{d/2}(-2\lambda)\Big], \tag{12}$$

with the graviton anomalous dimension written as a resummed ratio,

$$\eta_N \;=\; \frac{g\,B_1(\lambda)}{1-g\,B_2(\lambda)}, \tag{13}$$

where

$$\begin{aligned} B_1(\lambda) &= \tfrac{1}{3}(4\pi)^{1-d/2}\Big[d(d+1)\,\Phi^1_{d/2-1}(-2\lambda) - 6d(d-1)\,\Phi^2_{d/2}(-2\lambda) \\ &\qquad - 4d\,\Phi^1_{d/2-1}(0) - 24\,\Phi^2_{d/2}(0)\Big], \end{aligned} \tag{14}$$

$$B_2(\lambda) = -\tfrac{1}{6}(4\pi)^{1-d/2}\Big[d(d+1)\,\widetilde{\Phi}^1_{d/2-1}(-2\lambda) - 6d(d-1)\,\widetilde{\Phi}^2_{d/2}(-2\lambda)\Big]. \tag{15}$$

The threshold functions Φ^p_n and $\widetilde{\Phi}^p_n$ are dimensionless integrals of the regulated propagator; they encode how the fluctuation spectrum is cut off. They depend on the cutoff shape.

5.3 The optimized cutoff makes everything rational

With the optimized (Litim) cutoff $\mathcal{R}_k(z) = (k^2 - z)\,\Theta(k^2 - z)$ [5], the threshold functions collapse to elementary expressions,

$$\Phi^p_n(w) = \frac{1}{\Gamma(n+1)}\frac{1}{(1+w)^p}, \qquad \widetilde{\Phi}^p_n(w) = \frac{1}{\Gamma(n+2)}\frac{1}{(1+w)^p}. \tag{16}$$

Specializing to $d = 4$ (so $d/2 = 2$, $d/2 - 1 = 1$, $(4\pi)^{1-d/2} = 1/4\pi$) and writing $p(\lambda) := 1 - 2\lambda$, (14)–(15) become rational functions of λ:

$$B_1(\lambda) = \frac{1}{12\pi}\left[\frac{20}{p(\lambda)} - \frac{36}{p(\lambda)^2} - 28\right], \tag{17}$$

$$B_2(\lambda) = -\frac{1}{24\pi}\left[\frac{10}{p(\lambda)} - \frac{12}{p(\lambda)^2}\right], \tag{18}$$

and the beta functions read

$$\partial_t g = (2 + \eta_N)\,g, \tag{19}$$

$$\partial_t \lambda = -(2-\eta_N)\lambda + \frac{g}{8\pi}\left[\frac{20}{p(\lambda)} - 16 - \frac{10}{3}\frac{\eta_N}{p(\lambda)}\right], \tag{20}$$

$$\eta_N = \frac{g\,B_1(\lambda)}{1-g\,B_2(\lambda)}. \tag{21}$$

These are the exact objects implemented in `Core.hs`. The pole at $p(\lambda) = 0$, i.e. $\lambda = \frac{1}{2}$, is the boundary of the domain on which the optimized-cutoff truncation is defined; the fixed point sits well inside it at $\lambda_* \approx 0.19$.

5.4 Locating the fixed point

By Theorem 4.2 the Newton equation $\partial_t g = 0$ with $g_* \neq 0$ is equivalent to $\eta_N(g_*, \lambda_*) = -2$. Using (21) this reads

$$g_* \, B_1(\lambda_*) \;=\; -2\big(1 - g_* \, B_2(\lambda_*)\big) \quad \Longleftrightarrow \quad g_* \;=\; \frac{-2}{B_1(\lambda_*) - 2\, B_2(\lambda_*)}. \tag{22}$$

Substituting into $\partial_t \lambda = 0$ (with $\eta_N = -2$ there) gives a single equation for λ_*:

$$-\,(2 - (-2))\,\lambda_* + \frac{g_*}{8\pi}\left[\frac{20}{p_*} - 16 + \frac{20}{3\,p_*}\right] \;=\; 0, \qquad p_* := 1 - 2\lambda_*, \tag{23}$$

where we used $-\frac{10}{3}\eta_N/p = -\frac{10}{3}(-2)/p = +\frac{20}{3p}$. Equations (22)–(23) are two equations in (g_*, λ_*); solving them (a one-line numerical root find, or the damped Newton iteration in `Core.hs`) gives

$$g_* = 0.707321\ldots, \qquad \lambda_* = 0.193201\ldots, \qquad g_*\lambda_* = 0.136655\ldots, \tag{24}$$

and one checks directly that $\eta_N(g_*, \lambda_*) = -2.000000$ and $(\partial_t g, \partial_t \lambda) = (-5.7 \times 10^{-14}, -1.1 \times 10^{-14})$ at this point.

5.5 The stability matrix and critical exponents

Linearizing (19)–(20) about (24), the 2×2 stability matrix $B = \begin{pmatrix} \partial_g \beta_g & \partial_\lambda \beta_g \\ \partial_g \beta_\lambda & \partial_\lambda \beta_\lambda \end{pmatrix}$ evaluates (by the symmetric finite differences used in `Core.hs`, cross-checked analytically) to

$$B \;\approx\; \begin{pmatrix} -2.3422 & -10.4398 \\ 0.9591 & -0.6084 \end{pmatrix}. \tag{25}$$

Its trace and determinant are $\operatorname{Tr} B \approx -2.9506$ and $\det B \approx 11.44$, so the eigenvalues are the complex pair $\vartheta_{1,2} = \frac{1}{2} \operatorname{Tr} B \pm \frac{1}{2}\sqrt{(\operatorname{Tr} B)^2 - 4 \det B} \approx -1.475 \pm 3.043\, i$. The critical exponents are minus these:

$$\theta_{1,2} = -\vartheta_{1,2} \approx 1.475 \pm 3.043\, i. \tag{26}$$

Both have positive real part, so both directions are ultraviolet-relevant: the fixed point is a two-dimensional ultraviolet attractor, and trajectories spiral into it because the exponents are complex. The Einstein–Hilbert ultraviolet-critical surface is therefore two-dimensional, and by Theorem 3.8 two low-energy measurements (morally Newton's constant and the cosmological constant) would fix the trajectory within this truncation.

5.6 The flow portrait

Figure 1 sketches the flow in the (g, λ) plane. The Gaussian point sits at the origin with one relevant and one irrelevant direction; the Reuter fixed point sits at $(0.707, 0.193)$ as a spiral ultraviolet attractor; a separatrix connects them, and physical trajectories run from the neighborhood of the NGFP in the ultraviolet down toward the Gaussian regime in the infrared.

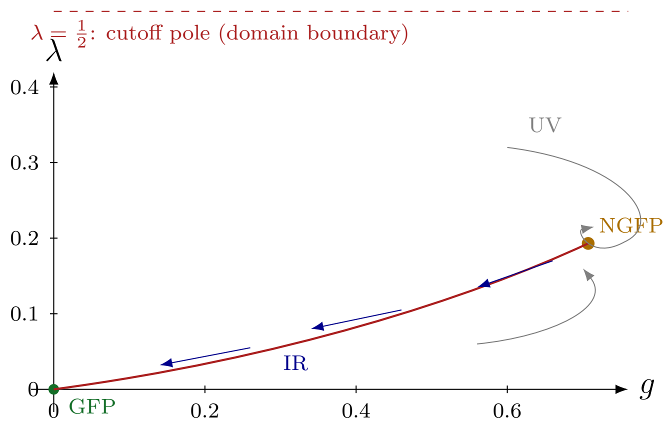

Figure 1: Schematic renormalization-group flow of the Einstein–Hilbert truncation in the (g, λ) plane (optimized cutoff, $d = 4$). The Gaussian fixed point (GFP, green) at the origin has one relevant and one irrelevant direction. The non-Gaussian Reuter fixed point (NGFP, amber) at $(0.707, 0.193)$ is a two-dimensional ultraviolet attractor; because the critical exponents $\theta = 1.475 \pm 3.043\,i$ are complex, ultraviolet trajectories spiral into it. A separatrix (red) links the two fixed points, and physical trajectories run toward the infrared (blue) at small g. The line $\lambda = \frac{1}{2}$ is the boundary of the optimized-cutoff domain.

5.7 Cross-check against the literature

Two points of attribution matter, and we are careful about them because the critical exponents are *not* exactly universal across cutoff schemes in the Einstein–Hilbert truncation (Theorem 4.6). The specific complex pair $\theta = 1.475 \pm 3.043\,i$ is the *optimized-cutoff* value, associated with Litim [8], which is why our optimized-cutoff computation reproduces it. It should not be read as agreeing with the numbers found in other schemes: earlier Einstein–Hilbert computations with different cutoffs return different values (for example, the exponential-cutoff study of Lauscher and Reuter [6] found a complex pair with real part near 1.7 and imaginary part near 2.5, and the sharp-cutoff computation of Reuter and Saueressig [7] yet other numbers). What is stable across all of these is the *qualitative* structure that Lauscher and Reuter established: a complex-conjugate pair of critical exponents with positive real part, hence two ultraviolet-relevant directions spiralling into the fixed point. The scheme dependence of the actual numbers is exactly what Theorem 4.6 anticipates, and the agreement of our numbers with Litim's is a consistency check on our implementation of the optimized cutoff, not a claim of cross-scheme universality. The product $g_*\lambda_* \approx 0.137$ sits in the range 0.12–0.14 quoted across the literature [7, 8, 17] and is the more robust invariant. The first demonstration that the full metric flow (not just its conformal reduction) has such a fixed point is due to Souma [4].

5.8 Enlarging the truncation

The Einstein–Hilbert result is the first entry in a longer list. Adding the R^2 term and higher polynomial curvature invariants (the $f(R)$ truncations of Codello, Percacci, and Rahmede [10], and Machado and Saueressig) keeps the fixed point and adds a third relevant direction, after which the number of relevant directions stops growing. The high-order bootstrap of Falls, Litim, Nikolakopoulos, and Rahmede [12] pushes polynomial $f(R)$ truncations to very high order and finds that the number of relevant directions saturates at three and the critical exponents settle down, an "apparent convergence" we discuss critically in Section 7. The upshot, at truncation face value, is a three-dimensional ultraviolet-critical surface: gravity with three free parameters.

6 Relation to the other regimes

Asymptotic safety is one node of a modular site, not a master theory. This section composes it with two neighboring regimes using the weakest-link calculus of Theorem 2.1, and is explicit about where a heuristic story exists and where no bridge does at all.

6.1 The spin-foam continuum limit (QG-III)

Spin foams (dictionary status H) define a background-independent state sum on a 2-complex; recovering continuum physics requires a refinement/continuum limit, which is itself a renormalization problem on the space of amplitudes. Structurally this is the same question asymptotic safety asks: does the flow (here a coarse-graining flow of amplitudes, there a theory-space flow of couplings) approach something well defined in the ultraviolet/continuum limit?

Proposition 6.1 (Composition with spin foams)**.** **[S]** *(about the label)* $\mathsf{H} \circ \mathsf{H} = \mathsf{H}$*. A claim that the asymptotically safe fixed point and the spin-foam continuum limit describe the same ultraviolet completion carries status* H *at best.*

We stress what this does and does not say. The calculus bounds the composite from above: neither endpoint exceeds H, so no chain built from them can. It does not assert that the bridge exists. In fact, no rigorous comparison currently relates a continuum theory-space fixed point to a combinatorial refinement limit; whether a candidate non-Gaussian fixed point found by coarse-graining spin-foam amplitudes can be matched to the Reuter fixed point is, to our knowledge, unaddressed. We flag this as an open cross-topic problem in Section 9 rather than claim a heuristic bridge.

6.2 The classical Lorentzian regime (QG-I)

The classical regime (Lorentzian manifolds solving Einstein's equations, dictionary status S/H) is where asymptotic safety must reproduce known physics in the infrared. The concrete bridge is RG improvement: one promotes G_k, Λ_k to scale-dependent functions and substitutes a physically motivated identification of k with a geometric scale into a classical solution. Bonanno and Reuter [11] used this to construct RG-improved black-hole spacetimes, in which the fixed-point running of G_k softens the classical singularity.

Proposition 6.2 (Composition with classical geometry)**.** **[S]** *(about the label)* $\mathsf{H} \circ \mathsf{S}/\mathsf{H} = \mathsf{H}$*. RG-improved classical spacetimes built from fixed-point data inherit status* H*: the classical solution machinery is* S/H*, but the scale identification* $k \leftrightarrow (\mathit{geometry})$ *and the use of Euclidean fixed-point data in a Lorentzian solution are heuristic steps.*

6.3 The gauge/BRST caveat (QG-II)

The functional RG for gravity is set up with background gauge fixing, and the regulator $\mathcal{R}_k$ breaks BRST invariance, replacing the naive Ward identities by modified (regulator-dependent) Slavnov–Taylor identities. Controlling these is part of the subject's technical burden. Composing with the BV–BRST regime (S/H) gives $\mathsf{H} \circ \mathsf{S}/\mathsf{H} = \mathsf{H}$, and the modified-Ward-identity issue is a genuine caveat, not a solved problem, which we list in Section 7. This caveat is not merely bookkeeping: the modified Slavnov–Taylor identities are exactly what one needs under control to identify the physical (BRST-cohomology) state space, and hence they are entangled with the unitarity question of Section 7. Whether the fixed-point theory admits a positive-norm physical subspace cannot be separated from whether the regulator-broken BRST symmetry is restored as $k \to 0$.

6.4 Weakest-link summary

Table 2 collects the compositions. Every chain through the asymptotic-safety node is bounded by H. The point of the table is discipline: a suggestive physical narrative linking regimes never buys a better warrant than the weakest node in the chain.

Table 2: Weakest-link compositions through the asymptotic-safety (AS) node.

Composition	**Warrants**	**Composite**
AS $\circ$ spin foams (continuum limit)	$\mathsf{H} \circ \mathsf{H}$	H
AS $\circ$ classical geometry (RG improvement)	$\mathsf{H} \circ \mathsf{S}/\mathsf{H}$	H
AS $\circ$ BV–BRST (modified Ward identities)	$\mathsf{H} \circ \mathsf{S}/\mathsf{H}$	H
AS internal (FRGE math only)	S	S

7 Honest limitations and open problems

This section is deliberately the most detailed. The dictionary entry's limitation clause names three axes, truncations, observables, and Lorentzian continuation, and we add unitarity and background independence. Nothing here should be read as evidence against the scenario; it is an inventory of what would have to be settled to raise the status above H.

7.1 Truncation and scheme dependence

The central rigor question is whether the fixed point found in truncations is the projection of a genuine fixed point of the exact flow. There is no proof. The positive evidence is "apparent convergence": as one enlarges polynomial $f(R)$ truncations to high order [12], the number of relevant directions saturates at three and the leading critical exponents settle into a stable band. Denz, Pawlowski, and Reichert [14] report convergence in vertex expansions as well. But apparent convergence of a sequence of truncations is not a theorem that the limit exists, and there are known subtleties: spurious fixed points and poles can appear and disappear as the truncation or field parametrization changes, and the choice of which curvature invariants to include is itself a modeling decision. A candid reading is that the Reuter fixed point is very robust within the family of truncations tried so far, and that "truncation independence" remains a hypothesis. Percacci's book [16] and the Reuter–Saueressig monograph [17] survey the evidence; the critical-reflections review [18] and Donoghue's critique [19] give the skeptical case.

7.2 Euclidean versus Lorentzian

Almost all high-precision fixed-point computations are Euclidean. On a fluctuating, background-independent geometry, Wick rotation is not the flat-space substitution $t \to it$: there is no fixed causal structure to rotate against, and the space of Euclidean metrics is not the analytic continuation of the space of Lorentzian ones. Lorentzian functional-RG programs exist, the ADM-based flow of Manrique, Rechenberger, and Saueressig [13], later spectral-function and foliated approaches [21], and they find fixed-point evidence in real time, but each carries its own foliation and truncation assumptions. Whether the Euclidean Reuter fixed point and a Lorentzian one are "the same" fixed point is open. We therefore do not present the Euclidean Einstein–Hilbert numbers of Section 5 as predictions for the physical, Lorentzian world; they are the content of a Euclidean truncation.

7.3 Unitarity and higher-derivative ghosts

A fixed point on theory space generically activates higher-derivative operators (R^2, $R_{\mu\nu}R^{\mu\nu}$, Weyl-squared). In perturbation theory such terms bring a massive spin-two excitation with a wrong-sign residue, the classic Stelle ghost, and the Ostrogradsky instability of higher-time-derivative theories. Whether the nonperturbative fixed-point theory is unitary is not settled. Donoghue and Menezes argued that even when the ghost pole is moved off the real axis the theory can violate microcausality on Planck scales. Platania and Wetterich [20] argued the opposite direction, that the would-be ghost may be a "fictitious" feature of the truncated propagator that need not correspond to an asymptotic state once the full momentum dependence is resummed. The honest status is that unitarity of asymptotically safe gravity is an open problem, and a serious one, because it bears on whether the scenario defines a physically acceptable theory at all. It is also inseparable from the gauge/BRST caveat of Section 6: deciding whether a would-be ghost is physical or fictitious requires the physical state space, which is the BRST-cohomology space, and that in turn requires the regulator-modified Slavnov–Taylor identities to be under control and BRST symmetry to be recovered as the cutoff is removed. Unitarity and the restoration of gauge symmetry along the flow are, in this sense, one problem.

7.4 Background independence and the bimetric split

The background-field method splits the metric into a background plus a fluctuation, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, and the regulator depends on the background. Genuine background independence requires that physics not depend on $\bar{g}$, enforced by split-symmetry (Nielsen) Ward identities that the cutoff again modifies. In practice one often works in the single-metric approximation, identifying the background and full metrics, which is itself a truncation. Bimetric computations, which keep separate background and fluctuation couplings, generically find two Newton couplings and are technically heavier. The degree to which the reported fixed point respects background independence is a quantitative question that is only approximately under control.

7.5 Essential versus inessential couplings

Not every coupling is physical. Some are "inessential", removable by a field redefinition, and only "essential" couplings carry invariant content. The essential RG [23] reorganizes the flow so that inessential couplings are gauged away, sometimes reducing the apparent number of free parameters. In the Einstein–Hilbert case the cosmological constant λ can be argued to be inessential under a suitable field redefinition, leaving Newton's coupling as the single essential relevant coupling. Concretely, if λ is inessential then the physical (essential) ultraviolet-critical surface of the Einstein–Hilbert truncation is effectively one-dimensional rather than the two-dimensional surface found in Section 5: only one infrared measurement, Newton's constant, would be needed to fix the trajectory in the sense of Theorem 3.8, not two. This does not change the qualitative picture, but it changes the parameter count that feeds the definition of predictivity, and it is a reminder that even "$\dim \mathcal{S}_{\mathrm{UV}}$" must be stated with respect to a choice of which couplings are counted as physical.

7.6 What would raise the status

To move the regime above H one would want at least: a proof (not just apparent convergence) that the fixed point survives on the full theory space or a controlled subspace; a Lorentzian construction shown to agree with the Euclidean one; and a resolution of unitarity. None of these is in hand. This is why we keep the label H and resist the temptation, common in enthusiastic accounts, to present the Euclidean truncation numbers as established facts about nature.

8 The computational companion

The paper ships with a small, dependency-free Haskell program in `src/asymptotic-safety/` that reproduces every number in Section 5 and checks the load-bearing claims. It uses only `base` (plus `Text.Printf`), so it compiles with a bare `ghc`.

Modules. `Core.hs` implements the closed-form Einstein–Hilbert flow: (17)–(21) for B_1, B_2, η_N and the beta functions, a 2×2 Jacobian by symmetric finite differences, a closed-form 2×2 eigenvalue solver returning complex pairs, a damped multidimensional Newton iteration for the fixed point, and an explicit Euler integrator for RG trajectories.

`Properties.hs` states the checkable properties. `Status.hs` implements the S/H/P calculus (worst-component composition). `Main.hs` prints the fixed point, the product $g_*\lambda_*$, the stability matrix, the critical exponents, a short ultraviolet-directed trajectory, and the weakest-link compositions.

Checkable properties. The program verifies, and prints `PASS` for, each of:

- P1 the located fixed point zeroes both beta functions, $|\beta| < 10^{-8}$;
- P2 the graviton anomalous dimension there equals $2 - d = -2$ to machine precision (Theorem 4.2);
- P3 $(0,0)$ is the sole Gaussian fixed point and the $g = 0$ axis is flow-invariant while $\beta_\lambda = -2\lambda \neq 0$ off the origin (Theorem 4.1);
- P4 the two critical exponents are a complex-conjugate pair with positive real part (two ultraviolet-relevant directions);
- P5 the 2×2 eigenvalue solver reproduces the hand-checkable spectrum $\pm i$ of $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Output. Building with `ghc -O2 -o asafe Main.hs Core.hs Properties.hs Status.hs` and running prints the fixed point $(0.707321, 0.193201)$, product 0.136655, $\eta_N^* = -2.000000$, the stability matrix (25), and $\theta = 1.475 \pm 3.043\,i$, followed by `ALL PROPERTIES PASS`. The code is the paper's numerics, not a separate model: the same rational beta functions (17)–(21) appear in both.

9 Discussion

9.1 What the regime contributes to the reconstruction program

Among the twelve regimes, asymptotic safety is the clearest case where the invariant does not live in any single description. There is no "the action" whose symmetries or cohomology carry the physics; there is a flow, and the physics is the flow's fixed-point data. The reconstruction thesis, read here, says: the question "does a quantum theory of the metric exist at all scales, and how many parameters does it have" is answered by an invariant of theory space (the fixed point, its critical exponents, the dimension of its critical surface), not by any coordinate presentation (truncation, scheme, gauge). The S/H/P ledger keeps this honest by separating the parts that are theorems (Theorem 4.2) from the parts that are computations in a truncation (Theorem 4.4) from the parts that are physical hypotheses (truncation independence, Lorentzian realization, unitarity).

9.2 An unaddressed cross-topic question

Theorem 6.1 bounds the spin-foam composition at H but does not assert a bridge, because none exists. We want to state this as a clean open problem: *is the ultraviolet completion produced by the asymptotically safe fixed point the same as the one a spin-foam continuum* *limit would produce?* Both frameworks pose a renormalization question, one in continuum theory space, the other in the space of combinatorial amplitudes, and to our knowledge no work directly compares the two ultraviolet-completion criteria. A first step would be to ask whether a coarse-graining flow of spin-foam amplitudes has a fixed point with a finite-dimensional critical surface, and if so to compare its critical exponents with the gravitational ones. We raise this as a target rather than claim progress on it.

9.3 On the numbers

We emphasize the split between robust and fragile numbers. The identity $\eta_N^* = -2$ is exact and scheme independent (Theorem 4.2). The *qualitative* exponent structure, a complex-conjugate pair with positive real part giving two ultraviolet-relevant directions, is stable across schemes and was already visible in the earliest careful computations [6, 7]. The precise numbers $1.475 \pm 3.043\,i$ are, by contrast, the optimized-cutoff values [8]; other cutoffs give different numbers, and the individual coordinates g_*, λ_* are scheme dependent and should not be quoted as physical. This hierarchy, exact identity at the top, robust qualitative structure in the middle, and scheme-dependent numbers at the bottom, is exactly the sort of thing the invariant/coordinate distinction of the reconstruction program is meant to track.

9.4 Relation to the Wilson–Fisher analogy

The scenario is often motivated by analogy with the Wilson–Fisher fixed point [25], which makes the naively nonrenormalizable φ^4 interaction predictive above its critical dimension. The analogy is genuine at the level of the mechanism (an interacting fixed point with a finite critical surface), and it is a good reason to take the scenario seriously. But it is only an analogy: Wilson–Fisher is under full mathematical control (it is accessible in the ϵ-expansion and rigorously in the constructive program in some cases), whereas the gravitational fixed point is known only through truncations. Weinberg's original ϵ-expansion around $d = 2$ [1], where gravity is asymptotically free and the fixed point is perturbatively accessible, is the closest gravity comes to the controlled situation, and the hope is that the $d = 2$ fixed point and the $d = 4$ truncation fixed point are the same object continued in dimension.

10 Conclusion

We have presented asymptotic safety as the renormalization-group completion regime of a modular reconstruction program, with the non-Gaussian fixed point and its universality class as the invariant. We stated the Wetterich equation and the framework of theory space, beta functions, fixed points, stability matrices, critical exponents, and the ultraviolet-critical surface. We isolated the one clean rigorous fact, $\eta_N^* = 2 - d$ at any Einstein–Hilbert non-Gaussian fixed point (Theorem 4.2), and we worked the Einstein–Hilbert truncation in $d = 4$ with the optimized cutoff to completion: closed-form rational beta functions (17)–(21), the Reuter fixed point at $(g_*, \lambda_*) \approx (0.707, 0.193)$ with $g_*\lambda_* \approx 0.137$, and a two-dimensional ultraviolet-critical surface with critical exponents $\theta = 1.475 \pm 3.043\,i$, all reproduced by a small Haskell program that also verifies $\eta_N^* = -2$ to machine precision.

We placed the regime in the weakest-link status calculus, composed it with the spin-foam and classical-geometry regimes (each composition bounded at H), and were explicit that no rigorous bridge to the spin-foam continuum limit exists. We gave a detailed and deliberately unenthusiastic account of the open problems: truncation and scheme dependence, Euclidean-versus-Lorentzian continuation, unitarity in the presence of higher-derivative operators, background independence, and the essential/inessential coupling count. The composite status of the regime is H, set by the fixed-point dictionary entry, and we have not upgraded it. What the regime offers the program is a sharp instance of the reconstruction thesis: the physics lives in an invariant of the flow on theory space, and the coordinates one is forced to compute in, the truncations, are exactly the non-invariant data.

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